ELEMENTS

OF

ALGEBRA:

INCLUDING STURMS' THEOREM.

TRANSLATED FROM THE FRENCH OF

M. BOURDON.

ADAPTED TO THE COURSE OF MATHEMATICAL INSTRUCTION IN THE UNITED STATES,

BY CHARLES DAVIES, LL.D.

AUTHOR OF ARITHMETIC, ELEMENTARY ALGEBRA, ELEMENTARY GEOMETRY, PRACTICAL GEOMETRY, ELEMENTS OF SURVEYING, ELEMENTS OF DESCRIPTIVE AND ANALYTICAL GEOMETRY, ELEMENTS OF DIFFERENTIAL AND INTEGRAL CALCULUS, AND A TREATISE ON SHADES, SHADOWS, AND PERSPECTIVE.

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PREFACE

The Treatise on Algebra, by M. Bourdon, is a work of singular excellence and merit. In France, it is one of the leading text books. Shortly after its first publication, it passed through several editions, and has formed the basis of every subsequent work on the subject of Algebra.

The original work is, however, a full and complete treatise on the subject of Algebra, the later editions containing about eight hundred pages octavo. The time which is given to the study of Algebra, in this country, even in those seminaries where the course of mathematics is the fullest, is too short to accomplish so voluminous a work, and hence it has been found necessary either to modify it, or to abandon it altogether.

The following work is abridged from a translation of M. Bourdon, made by Lieut. Ross, now the distinguished professor of mathematics in Kenyon College, Ohio.

The Algebra of M. Bourdon, however, has been regarded only as a standard or model. The order of arrangement, in many parts, has been changed; new rules and new methods have been introduced; and all the modifications which have

been suggested by teaching and a careful comparison with other standard works, have been freely made. It would, perhaps, not be just to regard M. Bourdon as responsible for the work in its present form.

It has been the intention to unite in this work, the scientific discussions of the French, with the practical methods of the English school; that theory and practice, science and art, may mutually aid and illustrate each other.

CHARLES DAVIES.

WEST POINT, *June*, 1844.

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ELEMENTS

OF

ALGEBRA.

CHAPTER I.

PRELIMINARY DEFINITIONS AND REMARKS.

1. QUANTITY is a general term applied to everything which can be estimated or measured.

2. MATHEMATICS is the science which treats of the properties and relations of quantities.

3. ALGEBRA is that branch of mathematics in which the quantities considered are represented by letters, and the operations to be performed upon them are indicated by signs. The letters and signs are called *symbols*.

4. The sign $+$, is called *plus*, and when placed between two quantities indicates that they are to be added together. Thus, $9 + 5$ is read, 9 plus 5, and indicates that the quantity represented by 5 is to be added to the quantity represented by 9.

In like manner, $a + b$ is read, a plus b, and denotes that the quantity represented by b is to be added to the quantity represented by a.

5. The sign $-$, is called *minus*, and indicates that one quantity is to be subtracted from another. Thus, $9 - 5$ is read, 9 minus 5 or 9 diminished by 5.

In like manner, $a - b$ is read, *a minus b*, or a diminished by b

6. The sign $\times$, is called the sign of multiplication. When placed between two quantities, it denotes that they are to be multiplied together. Thus, 36×25, denotes that 36 is to be multiplied by 25. The multiplication of two quantities may also be

indicated by placing a point between them. Thus, 36.25 is the same as 36×25, and is read, 36 multiplied by 25, or the product of 36 by 25.

7. The multiplication of quantities, which are represented by letters, is generally indicated by simply writing the letters one after the other, without interposing any sign. Thus,

ab is the same as $a \times b$, or as $a.b$;

and abc, the same as $a \times b \times c$, or as $a.b.c$.

It is plain that the notation ab, or abc, cannot be employed when the quantities are represented by figures. For, if it were required to express the product of 5 by 6, we could not write 5 6, without confounding the product with the number 56.

8. In the product of several letters, as abc, the single letters, a, b, and c, are called *factors*. Thus, in the product ab, there are two factors, a and b; in the product acd, there are three, a, c, and d.

9. There are three signs used to denote division. Thus,

$a \div b$ denotes that a is to be divided by b.

$\frac{a}{b}$ denotes that a is to be divided by b.

$a\underline{|b}$ denotes that a is to be divided by b.

10. The sign $=$, is called the sign of equality, and is read, *is equal to*. When placed between two quantities, it denotes that they are equal to each other. Thus, $9 - 5 = 4$: that is, 9 minus 5 is equal to 4: Also, $a + b = c$, indicates that the sum of the quantities represented by a and b, is equal to the quantity denoted by c.

11. The sign $>$, is called the sign of *inequality*, and is used to express that one quantity is greater or less than another.

Thus, $a > b$ is read, a greater than b; and $a < b$ is read, a less than b; that is, the opening of the sign is turned toward the greater quantity.

12. If a quantity is added to itself several times, as

$$a + a + a + a + a,$$

it is generally written but once, and a figure is then placed before it, to show how many times it is taken. Thus,

$$a + a + a + a + a = 5a.$$

The number 5 is called the *co-efficient* of a, and denotes that a is taken 5 times.

Hence, a *co-efficient* is a number prefixed to a quantity, denoting the number of times which the quantity is taken. The co-efficient also indicates the number of times *plus one*, that the quantity is added to itself. When no co-efficient is written, the co-efficient 1 is always understood. Thus, $a = 1a$.

13. If a quantity be multiplied continually by itself, as

$$a \times a \times a \times a \times a,$$

the product is generally expressed by writing the letter once, and placing a number to the right of, and a little above it: thus,

$$a \times a \times a \times a \times a = a^5.$$

The number 5 is called the *exponent* of a, and denotes the number of times which a enters into the product as a factor.

Hence, the *exponent* of a quantity shows how many times the quantity is a factor. It also indicates the number of times, *plus one*, that the quantity is to be multiplied by itself. When no exponent is written, the exponent 1 is always understood.

14. The product resulting from the multiplication of a quantity by itself any number of times, is called the *power* of that quantity: and the exponent denotes the *degree* of the power. For example,

$a^1 = a$ is the first power of a,
$a^2 = a \times a$ is the second power, or square of a,
$a^3 = a \times a \times a$ is the third power, or cube of a,
$a^4 = a \times a \times a \times a$ is the fourth power of a,
$a^5 = a \times a \times a \times a \times a$ is the fifth power of a,

in which the exponents of the powers are, 1, 2, 3, 4, and 5; and the powers themselves, are the results of the multiplications. It should be observed that the *exponent of a power* is always greater by *unity* than the number of multiplications.

15. As an example of the use of the exponent in algebra, let it be required to express that a number a is to be multiplied three times by itself, that this product is then to be multiplied three times by b, and this new product twice by c; we should write

$$a \times a \times a \times a \times b \times b \times b \times c \times c = a^4b^3c^2.$$

If it were further required to repeat this result a certain num-

ber of times, say seven times, that is, *to add it to itself six times,* we should simply write

$$7a^4b^3c^2.$$

This example shows the brevity of the algebraic language.

16. The root of a quantity, is a quantity which being multiplied by itself a certain number of times, will produce the given quantity.

The sign $\sqrt{}$, is called the radical sign, and when prefixed to a quantity, indicates that its root is to be extracted. Thus,

$\sqrt[2]{a}$ or simply $\sqrt{a}$ denotes the square root of a.

$\sqrt[3]{a}$ denotes the cube root of a.

$\sqrt[4]{a}$ denotes the fourth root of a.

The number placed over the radical sign is called the *index* of the root. Thus, 2 is the index of the square root, 3 of the cube root, 4 of the fourth root, &c.

17. The *reciprocal* of a quantity, is unity divided by that quantity. Thus,

$\frac{1}{a}$ is the reciprocal of a;

and, $\frac{1}{a+b}$ is the reciprocal of $a+b$.

18. Every quantity written in algebraic language, that is, with the aid of letters and signs, is called an *algebraic quantity*, or the *algebraic expression* of a quantity. Thus,

$3a$ is the algebraic expression of three times the quantity denoted by a;

$5a^2$ is the algebraic expression of five times the square of a;

$7a^3b^2$ is the algebraic expression of seven times the product of the cube of a by the square of b;

$3a - 5b$ is the algebraic expression of the difference between three times a and five times b;

$2a^2 - 3ab + 4b^2$ is the algebraic expression of twice the square of a, diminished by three times the product of a by b, augmented by four times the square of b.

19. A single algebraic expression, not connected with any other by the sign of addition or subtraction, is called a *monomial*, or simply, a *term*.

Thus, $3a$, $5a^2$, $7a^3b^2$, are monomials, or single terms.

20. An algebraic expression composed of two or more terms, separated by the sign $+$ or $-$, is called a *polynomial*.

For example, $3a - 5b$, and $2a^2 - 3cb + 4b^2$, are polynomials.

A polynomial composed of two terms, is called a *binomial;* and a polynomial of three terms is called a *trinomial.*

21. The *numerical value* of an algebraic expression, is the number obtained by giving a particular value to each letter which enters it, and performing the arithmetical operations indicated. This numerical value will depend on the particular values attributed to the letters, and will *generally* vary with them.

For example, the numerical value of $2a^3$, will be 54 if we make $a = 3$; for, $3^3 = 27$, and $2 \times 27 = 54$.

The numerical value of the same expression is 250 when we make $a = 5$; for, $5^3 = 125$, and $2 \times 125 = 250$.

X 22. It has been said, that the numerical value of an algebraic expression *generally* varies with the values of the letters which enter it: it does not, however, always do so. Thus, in the expression $a - b$, so long as a and b are increased or diminished by the same number, the value of the expression will not be changed.

For example, make $a = 7$ and $b = 4$: there results $a - b = 3$. Now, make $a = 7 + 5 = 12$, and $b = 4 + 5 = 9$, and there results, as before, $a - b = 12 - 9 = 3$.

23. Of the different terms which compose a polynomial, some are preceded by the sign $+$, and others by the sign $-$. The first are called *additive terms*, the others, *subtractive terms.*

When the first term of a polynomial is plus, the sign is generally omitted; and when no sign is written, it is always understood to be affected by the sign $+$.

24. The *numerical value* of a polynomial is not affected by changing the *order* of its terms, provided the signs of all the terms remain unchanged. For example, the polynomial

$$4a^3 - 3a^2b + 5ac^2 = 5ac^2 - 3a^2b + 4a^3 = - 3a^2b + 5ac^2 + 4a^3.$$

25. Each of the literal factors which compose a term, is called a *dimension* of the term; and the *degree* of a term is the number of these factors or dimensions. Thus,

$3a$ is a term of one dimension, or of the first degree.

$5ab$ is a term of two dimensions, or of the second degree.

$7a^3bc^2 = 7aaabcc$ is of six dimensions, or of the sixth degree.

In general, *the degree, or the number of dimensions of a term, is determined by taking the sum of the exponents of the letters which enter this term.* For example, the term $8a^2bcd^3$ is of the seventh degree, since the sum of the exponents, $2 + 1 + 1 + 3 = 7$.

26. A polynomial is said to be *homogeneous*, when all its terms are of the same degree. The polynomial

$3a - 2b + c$	is of the first degree and homogeneous.
$-4ab + b^2$	is of the second degree and homogeneous.
$5a^2c - 4c^3 + 2c^2d$	is of the third degree and homogeneous.
$8a^3 - 4ab + c$	is not homogeneous.

27. A vinculum or bar $\overline{\quad\quad}$, or a parenthesis ( ), is used to express that all the terms of a polynomial are to be considered together. Thus, $\overline{a + b + c} \times b$, or $(a + b + c) \times b$ denotes that the trinomial $a + b + c$ is to be multiplied by b; also,

$$\overline{a + b + c} \times \overline{c + d + f} \text{ or } (a + b + c) \times (c + d + f)$$

denotes that the trinomial $a + b + c$ is to be multiplied by the trinomial $c + d + f$.

When the parenthesis is used, the sign of multiplication is usually omitted. Thus, $(a + b + c) \times b$ is the same as $(a + b + c)\,b$

The bar is also sometimes placed vertically. Thus,

$$\left.\begin{array}{r} +a \\ +b \\ +c \end{array}\right| x \quad \text{is the same as } (a + b + c)\,x, \text{ or, } \overline{a + b + c} \times x.$$

28. Those terms of a polynomial which are composed of the same letters, affected with the same exponents, are called *similar terms.*

Thus, in the polynomial

$$7ab + 3ab - 4a^3b^2 + 5a^3b^2,$$

the terms $7ab$ and $3ab$, are similar, and so also are the terms $-4a^3b^2$ and $5a^3b^2$, the letters and exponents in each being the same. But in the binomial

$$8a^2b + 7ab^2,$$

the terms are not similar; for, although they are composed of the same letters, yet the same letters are not affected with the same exponents.

29. When a polynomial contains several similar terms, it may often be reduced to a simpler form.

Take the polynomial $4a^2b - 3a^2c + 7a^2c - 2a^2b$.

It may be written (Art. 24) $4a^2b - 2a^2b + 7a^2c - 3a^2c$.

But $4a^2b - 2a^2b$ reduces to $2a^2b$, and $7a^2c - 3a^2c$ to $4a^2c$.

Hence, $4a^2b - 3a^2c + 7a^2c - 2a^2b = 2a^2b + 4a^2c$.

When we have a polynomial having similar terms, as

$$+ 2a^3bc^2 - 4a^3bc^2 + 6a^3bc^2 - 8a^3bc^2 + 11a^3bc^2,$$

unite the additive and subtractive terms separately: thus,

Additive terms.	*Subtractive terms.*
$+ 2a^3bc^2$	$- 4a^3bc^2$
$+ 6a^3bc^2$	$- 8a^3bc^2$
$+ 11a^3bc^2$	$\overline{- 12a^3bc^2}$
$\overline{+ 19a^3bc^2}$	

Hence, the given polynomial reduces to

$$19a^3bc^2 - 12a^3bc^2 = 7a^3bc^2.$$

It may happen that the co-efficient of the subtractive term, obtained as above, will exceed that of the additive term. In that case, *subtract the positive co-efficient from the negative, prefix the minus sign to the remainder, and then annex the literal part.*

In the polynomial $3a^2b + 2a^2b - 5a^2b - 3a^2b$

$+ 3a^2b$	$- 5a^2b$
$+ 2a^2b$	$- 3a^2b$
$\overline{+ 5a^2b}$	$\overline{- 8a^2b}$

But, $- 8a^2b = - 5a^2b - 3a^2b$: hence

$$5a^2b - 8a^2b = 5a^2b - 5a^2b - 3a^2b = - 3a^2b.$$

Hence, for the reduction of the similar terms of a polynomial, we have the following

RULE.

I. *Add together the co-efficients of all the additive terms, and annex to their sum the literal part: form a single subtractive term in the same manner.*

II. *Then, subtract the less co-efficient from the greater, and to the remainder prefix the sign of the greater co-efficient, and annex the literal part.*

EXAMPLES.

1. Reduce the polynomial $4a^2b - 8a^2b - 9a^2b + 11a^2b$ to its simplest form. *Ans.* $-2a^2b$.

2. Reduce the polynomial $7abc^2 - abc^2 - 7abc^2 - 8abc^2 + 6abc^2$ to its simplest form. *Ans.* $-3abc^2$.

3. Reduce the polynomial $9cb^3 - 8ac^2 + 15cb^3 + 8ca + 9ac^2 - 24cb^3$ to its simplest form. *Ans.* $ac^2 + 8ca$.

4. Reduce the polynomial $6ac^2 - 5ab^3 + 7ac^2 - 3ab^3 - 13ac^2 + 18ab^3$ to its simplest form. *Ans.* $10ab^3$.

5. Reduce the polynomial $abc^2 - abc + 5ac^2 - 9abc^2 + 6abc - 8ac^2$ to its simplest form. *Ans.* $-8abc^2 + 5abc - 3ac^2$.

REMARK.—It should be observed that the reduction affects only the co-efficients, and not the exponents.

The reduction of similar terms is an operation peculiar to algebra. Such reductions are constantly made in *Algebraic Addition, Subtraction, Multiplication,* and *Division.*

30. In the operations of algebra, there are two kinds of quantities which must be distinguished from each other, viz.

1st, Those whose values are known or given, and which are called *known quantities;* and

2dly, Those whose values are unknown, which are called *unknown quantities.*

The known quantities are represented by the first letters of the alphabet, a, b, c, d, &c.; and the unknown, by the final letters, x, y, z, &c.

31. A *problem* is a question proposed which requires a solution. It is said to be solved when the values of the quantities sought are discovered or found.

A *theorem* is a general truth, which is proved by a course of reasoning called a demonstration.

32. The following question will tend to show the utility of the algebraic analysis.

Question.

The sum of two numbers is 67, *and their difference* 19; *what are the two numbers?*

Solution.

Let us first establish, by the aid of the algebraic symbols, the connexion which exists between the given and unknown numbers of the question.

If the least of the two numbers were *known*, the greater could be found by adding to it the difference 19; or in other words, the less number, plus 19, is equal to the greater.

If, then, we make $x \quad = $ the less number,

$x + 19 =$ the greater,

and $2x + 19 =$ the sum.

But from the enunciation, this sum is to be equal to 67. Therefore we have

$$2x + 19 = 67.$$

Now, if $2x$ augmented by 19, gives 67, $2x$ alone is equal to 67 minus 19, or

$2x = 67 - 19$, or performing the subtraction, $2x = 48$.

Hence, x is equal to half of 48, that is,

$$x = \frac{48}{2} = 24.$$

The least number being 24, the greater is

$$x + 19 = 24 + 19 = 43.$$

And, indeed, we have

$$43 + 24 = 67, \text{ and } 43 - 24 = 19.$$

Another Solution.

Let x represent the greater number;

then, $x - 19$ will represent the less

and $2x - 19 = 67$; whence, $2x = 67 + 19 = 86$;

therefore, $x = \frac{86}{2} = 43 =$ the greater,

and consequently, $x - 19 = 43 - 19 = 24 =$ the less

General Solution of this Problem.

The sum of two numbers is a, and their difference is b. What are the two numbers?

Let $x \quad =$ the less number;

then will, $x + b =$ the greater.

Then, by the conditions of the question

$$2x + b = a, \text{ the sum of the numbers;}$$

therefore, $2x = a - b$ and $x = \frac{a-b}{2} = \frac{a}{2} - \frac{b}{2}$.

And by adding b to each side of the equality, we obtain the greater number,

$$x + b = \frac{a}{2} - \frac{b}{2} + b = \frac{a}{2} + \frac{b}{2}.$$

Hence we have

$$x + b = \frac{a}{2} + \frac{b}{2} = \text{the greater number,}$$

and

$$x = \frac{a}{2} - \frac{b}{2} = \text{the less.}$$

As the *form* of these results is independent of any particular values attributed to the letters a and b, the expressions are called *formulas*, and may be regarded as comprehending the solution of all questions of the same nature, differing only in the numerical values of the given quantities. Hence,

A formula is the algebraic enunciation of a general rule, or principle.

The principles enunciated by the formulas above, are these:

The greater of any two numbers is equal to half their sum increased by half their difference; and *the less, is equal to half their sum diminished by half their difference.*

To apply these formulas to the case in which the sum is 237 and difference 99, we have

$$\text{the greater number } = \frac{237}{2} + \frac{99}{2} = \frac{237 + 99}{2} = \frac{336}{2} = 168;$$

$$\text{and the less } = \frac{237}{2} - \frac{99}{2} = \frac{237 - 99}{2} = \frac{138}{2} = 69;$$

and these are the true numbers; for,

$$168 + 69 = 237 \text{ which is the given sum,}$$

and

$$168 - 69 = 99 \text{ which is the given difference.}$$

From the preceding explanations, we see that Algebra is a *language* composed of a series of symbols, by the aid of which, *we can abridge and generalize the operations required in the solution of problems, and the reasonings pursued in the demonstration of theorems.*

CHAPTER II.

OF ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION.

ADDITION.

33. ADDITION, in algebra, consists in finding the simplest equivalent expression for several algebraic quantities. Such equivalent expression is called their *sum*.

34. Let it be required to add together the monomials,

$$\left\{\begin{array}{c} 3a \\ 5b \\ 2c \end{array}\right.$$

The result of the addition is - - - $3a + 5b + 2c$

an expression which cannot be reduced to a more simple form.

Again, add together the monomials -

$$\left\{\begin{array}{c} 4a^2b^3 \\ 2a^2b^3 \\ 7a^2b^3 \end{array}\right.$$

The result, after reducing (Art. 29), is - $13a^2b^3$

Let it be required to find the sum of the expressions,

$$\left\{\begin{array}{r} 3a^2 - 4ab \qquad \\ 2a^2 - 3ab + b^2 \\ 2ab - 5b^2 \end{array}\right.$$

Their sum, after reducing (Art. 29), is - $5a^2 - 5ab - 4b^2$

35. As a course of reasoning similar to the above would apply to all algebraic expressions, we deduce, for the addition of algebraic quantities, the following general

RULE.

I. *Write down the quantities to be added, with their respective signs, so that the similar terms shall fall under each other.*

II. *Reduce the similar terms, and annex to the results those terms which cannot be reduced, giving to each term its respective sign.*

EXAMPLES.

1. Add together the polynomials,

$3a^2 - 2b^2 - 4ab,\quad 5a^2 - b^2 + 2ab,\quad$ and $\quad 3ab - 3c^2 - 2b^2.$

The term $3a^2$ being similar to $5a^2$, we write $8a^2$ for the result of the reduction of these two terms, at the same time slightly crossing them as in the terms of the example.

$$\begin{array}{rrrr} \cancel{3a^2} & -\cancel{4ab} & -\cancel{2b^2} & \\ \cancel{5a^2} & +\cancel{2ab} & -\cancel{b^2} & \\ & +\cancel{3ab} & -\cancel{2b^2} & -3c^2 \\ \hline 8a^2 & +\ ab & -5b^2 & -3c^2 \end{array}$$

Passing then to the term $-4ab$, which is similar to the two terms $+2ab$ and $+3ab$, the three reduce to $+ab$, which is placed after $8a^2$, and the terms crossed like the first term. Passing then to the terms involving b^2, we find their sum to be $-5b^2$, after which we write $-3c^2$.

The marks are drawn across the terms, that none of them may be overlooked and omitted.

(2).

$$\begin{array}{lrrr} & 7x & +3ab & +\ 2c \\ & -3x & -3ab & -\ 5c \\ & 5x & -9ab & -\ 9c \\ \hline \text{Sum.} & 9x & -9ab & -12c \end{array}$$

(3).

$$\begin{array}{rrr} 16a^2b^2 & +\ bc & -2abc \\ -4a^2b^2 & -9bc & +6abc \\ -9a^2b^2 & +\ bc & +\ abc \\ \hline 3a^2b^2 & -7bc & +5abc \end{array}$$

(4).

$$\begin{array}{rrrr} a & +\ ab & -\ cd & +\ f \\ 3a & +5ab & -6cd & -\ f \\ -5a & -6ab & +6cd & -7f \\ -\ a & +\ ab & +\ cd & +4f \\ \hline -2a & +\ ab & +0 & -3f \end{array}$$

(5).

$$\begin{array}{rrrr} 6ab & +\ cd & +d & \\ 3ab & +\ 5cd & +y & \\ -4ab & +\ 6cd & +x & \\ -5ab & -12cd & -y & \\ \hline 0 & 0 & +x & +d \end{array}$$

6. Add together $3a + b$, $3a + 3b$, $-9a - 7b$, $6a + 9b$, and $8a + 3b + 8c$. *Ans.* $11a + 9b + 8c$.

7. Add together $3ax + 3ac + f$, $-9ax + 7a + d$, $+6ax + 3ac + 3f$, $8ax + 13ac + 9f$, and $-14f + 3ax$.

Ans. $11ax + 19ac - f + 7a + d$.

8. Add together the polynomials, $3a^2c + 5ab$, $7a^2c - 3ab + 3ac$, $5a^2c - 6ab + 9ac$, and $-8a^2c + ab - 12ac$. *Ans.* $7a^2c - 3ab$.

9. Add the polynomials $19a^2x^3b - 12a^3cb$, $5a^2x^3b + 15a^3cb - 10ax$, $-2a^2x^3b - 13a^3cb$, and $-18a^2x^3b - 12a^3cb + 9ax$.

Ans. $4a^2x^3b - 22a^3cb - ax$

10. Add together $3a + b + c$, $5a + 2b + 3ac$, $a + c + ac$, and $-3a - 9ac - 8b$. *Ans.* $6a - 5b + 2c - 5ac$.

11. Add together $5a^2b + 6cx + 9bc^2$, $7cx - 8a^2b$, and $-15cx - 9bc^2 + 2a^2b$. *Ans.* $-a^2b - 2cx$.

12. Add together $8ax + 5ab + 3a^2b^2c^2$, $-18ax + 6a^2 + 10ab$, and $10ax - 15ab - 6a^2b^2c^2$. *Ans.* $-3a^2b^2c^2 + 6a^2$.

13. What is the sum of $41a^3b^2c - 27abc - 14a^2y$ and $10a^3b^2c + 9abc$? *Ans.* $51a^3b^2c - 18abc - 14a^2y$.

14. What is the sum of $18abc - 9ab + 6c^2 - 3c + 9ax$ and $9abc + 3c - 9ax$? *Ans.* $27abc - 9ab + 6c^2$.

15. What is the sum of $8abc + b^3a - 2cx - 6xy$ and $7cx - xy - 13b^3a$? *Ans.* $8abc - 12b^3a + 5cx - 7xy$.

16. What is the sum of $9a^2c - 14aby + 15a^2b^2$ and $-a^2c - 8a^2b^2$? *Ans.* $8a^2c - 14aby + 7a^2b^2$.

17. What is the sum of $17a^5b^2 + 9a^3b - 3a^2$, $-14a^5b^2 + 7a^2 - 9a^3$, $-15a^3b + 7a^5b^2 - a^3$, and $14a^3b - 19a^3b$?

Ans. ————.

18. What is the sum of $3ax^2 - 9ax^3 - 17axy$, $-9ax^2 + 18ax^3 + 34axy$, and $7a^5b + 3ax^3 - 7ax^2 + 4bcx$? *Ans.* ————.

19. Add together $3a^2 + 5a^2b^2c^2 - 9a^3x$, $7a^2 - 8a^2b^2c^2 - 10a^3x$, $10ab + 16a^2b^2c^2 + 19a^3x$. *Ans.* $10a^2 + 13a^2b^2c^2 + 10ab$.

20. Add together $7a^2b - 3abc - 8b^2c - 9c^3 + cd^2$, $8abc - 5a^2b + 3c^3 - 4b^2c + cd^2$, and $4a^2b - 8c^3 + 9b^2c - 3d^3$.

Ans. $6a^2b + 5abc - 3b^2c - 14c^3 + 2cd^2 - 3d^3$.

21. Add together $-18a^3b + 2ab^4 + 6a^2b^2 - 8ab^4 + 7a^3b - 5a^2b^2 - 5a^3b + 6ab^4 + 11a^2b^2$. *Ans.* $-16a^3b + 12a^2b^2$.

22. What is the sum of $3a^3b^2c - 16a^4x - 9ax^3d$, $+6a^3b^2c - 6ax^3d + 17a^4x$, $+16ax^3d - a^4x - 8a^3b^2c$?

Ans. $a^3b^2c + ax^3d$.

23. What is the sum of the following terms: viz., $8a^5 - 10a^4b - 16a^3b^2 + 4a^2b^3 - 12a^4b + 15a^3b^2 + 24a^2b^3 - 6ab^4 - 16a^3b^2 + 20a^2b^3 + 32ab^4 - 8b^5$?

Ans. $8a^5 - 22a^4b - 17a^3b^2 + 48a^2b^3 + 26ab^4 - 8b^5$.

SUBTRACTION.

36. SUBTRACTION, in Algebra, consists in finding the simplest expression for the difference between two algebraic quantities.

Let it be required to subtract $4b$ from $5a$. Here, as the quantities are not similar, their difference can only be indicated, and we write

$$5a - 4b.$$

Again, let it be required to subtract $4a^3b$ from $7a^3b$. These terms being similar, one of them may be taken from the other, and their true difference is expressed by

$$7a^3b - 4a^3b = 3a^3b.$$

For another example, from - - - $4a$
take the binomial - - - - $2b - 3c$.

The subtraction may be indicated thus, - $4a - (2b - 3c)$;

that is, the quantity to be subtracted may be placed within a parenthesis, and written after the other quantity, with a minus sign.

$4a - 2b + 3c$
True remainder.

Now, in order to express this difference by a single polynomial, let us see what the nature of the question requires.

From $4a$, we are to subtract the difference between $2b$ and $3c$, and if b and c were given numerically, that difference would be known; but since $3c$ cannot be taken from $2b$, $2b$ is first subtracted from $4a$, which gives $4a - 2b$. Now, in subtracting the number of units contained in $2b$, the number taken away from $4a$, is too great by the number of units contained in $3c$, and the result $4a - 2b$ is therefore too small by $3c$; this remainder must therefore be corrected by adding $3c$ to it. Hence, there results from the proposed subtraction $4a - 2b + 3c$.

37. Hence, for the subtraction of algebraic quantities, we have the following general

RULE.

I. *Write the quantity to be subtracted under that from which it is to be taken, placing the similar terms, if there are any, under each other.*

II. *Change the signs of all the terms of the polynomial to be subtracted, or conceive them to be changed, and then reduce the polynomial result to its simplest form*

EXAMPLES.

	(1).		(1).
From	$6ac - 5ab + c^2$	The same with the signs of the lower line changed.	$6ac - 5ab + c^2$
Take	$3ac + 3ab - 7c$		$-3ac - 3ab + 7c$
Remainder	$3ac - 8ab + c^2 + 7c.$		$3ac - 8ab + c^2 + 7c$

	(2).	(3).
From	$16a^2 - 5bc + 7ac$	$19abc - 16ax - 5axy$
Take	$14a^2 + 5bc + 8ac$	$17abc + 7ax - 15axy$
Remainder	$2a^2 - 10bc - ac$	$2abc - 23ax + 10axy$

	(4).	(5).
From	$5a^3 - 4a^2b + 3b^2c$	$4ab - cd + 3a^2$
Take	$-2a^3 + 3a^2b - 8b^2c$	$5ab - 4cd + 3a^2 + 5b^2$
Remainder	$7a^3 - 7a^2b + 11b^2c$	$-ab + 3cd + 0 - 5b^2.$

6. From $3a^2x - 13abc + 7a^2$, take $9a^2x - 13abc$.
Ans. $-6a^2x + 7a^2$.

7. From $51a^3b^2c - 18abc - 14a^2y$, take $41a^3b^2c - 27abc - 14a^2y$.
Ans. $10a^3b^2c + 9abc$.

8. From $27abc - 9ab + 6c^2$, take $9abc + 3c - 9ax$.
Ans. $18abc - 9ab + 6c^2 - 3c + 9ax$.

9. From $8abc - 12b^3a + 5cx - 7xy$, take $7cx - xy - 13b^3a$.
Ans. $8abc + b^3a - 2cx - 6xy$.

10. From $8a^2c - 14aby + 7a^2b^2$, take $9a^2c - 14aby + 15a^2b^2$.
Ans. $-a^2c - 8a^2b^2$.

11. From $9a^6x^2 - 13 + 20ab^3x - 4b^6cx^2$, take $3b^6cx^2 + 9a^6x^2 - 6 + 3ab^3x$.
Ans. $17ab^3x - 7b^6cx^2 - 7$.

12. From $5a^4 - 7a^3b^2 - 3c^5d^2 + 7d$, take $3a^4 - 3a^2 - 7c^5d^2 - 15a^3b^2$.
Ans. $2a^4 + 8a^3b^2 + 4c^5d^2 + 7d + 3a^2$.

13. From $51a^2b^2 - 48a^3b + 10a^4$, take $10a^4 - 8a^3b - 6a^2b^2$.
Ans. $57a^2b^2 - 40a^3b$.

14. From $21x^3y^2 + 25x^2y^3 + 68xy^4 - 40y^5$, take $64x^2y^3 + 48xy^4 - 40y^5$.
Ans. $20xy^4 - 39x^2y^3 + 21x^3y^2$.

15. From $53x^3y^2 - 15x^2y^3 - 18x^4y - 56x^5$, take $-15x^2y^3 + 18x^3y^2 + 24x^4y$.
Ans. $35x^3y^2 - 42x^4y - 56x^5$

38. From what has preceded, we see that polynomials may be subjected to certain transformations.

For example - -	$6a^2 - 3ab + 2b^2 - 2bc$,
may be written -	$6a^2 - (3ab - 2b^2 + 2bc)$.
In like manner - -	$7a^3 - 8a^2b - 4b^2c + 6b^3$,
may be written -	$7a^3 - (8a^2b + 4b^2c - 6b^3)$;
or, again, - - -	$7a^3 - 8a^2b - (4b^2c - 6b^3)$.
Also, - - - -	$8a^2 - 6a^2b^2 + 5a^2b^3$,
becomes - - -	$8a^2 - (6a^2b^2 - 5a^2b^3)$.
Also, - - - -	$9a^2c^3 - 8a^4 + b^2 - c$,
may be written -	$9a^2c^3 - (8a^4 - b^2 + c)$;
or, it may be written	$9a^2c^3 + b^2 - (8a^4 + c)$.

These transformations consist in decomposing a polynomial into two parts, separated from each other by the sign —.

It will be observed that the sign of each term which is placed within the parenthesis is changed. Hence, if we have one or more terms included within a parenthesis having the minus sign before it, if the parenthesis is omitted, *the signs of all the terms must be changed.*

Thus,	$4a - (6ab - 3c - 2b)$,
is equal to	$4a - 6ab + 3c + 2b$.
Also	$6ab - (-4ac + 3d - 4ab)$,
is equal to	$6ab + 4ac - 3d + 4ab$.

39. REMARK.—From what has been shown in addition and subtraction, we deduce the following principles.

1st. In Algebra, the words *add* and *sum* do not always, as in arithmetic, convey the idea of augmentation. For, $a - b$, which may result from the addition of $-b$ to a, is properly speaking, the arithmetical difference between the number of units expressed by a, and the number of units expressed by b. Consequently, this result is numerically less than a.

To distinguish this sum from an arithmetical sum, it is called the *algebraic sum.*

Thus, the polynomial, $2a^3 - 3a^2b + 3b^2c$.
is an algebraic sum, so long as it is considered as the result of the union of the monomials $2a^3$, $-3a^2b$, $+3b^2c$, with their respective signs; but, in its *proper acceptation*, it is the arithmeti-

cal difference between the sum of the units contained in the additive terms, and the sum of the units contained in the subtractive terms.

It follows from this, that an algebraic sum may, in the numerical applications, be reduced to a *negative* number, or a number affected with the sign $-$.

2d. The words *subtraction* and *difference*, do not always convey the idea of diminution. For, the difference between $+a$ and $-b$ being $a-(-b)=a+b$, is numerically greater than a. This result is an *algebraic difference*.

MULTIPLICATION.

40. ALGEBRAIC multiplication has the same object as arithmetical, viz., to repeat the multiplicand as many times as there are units in the multiplier. The multiplicand and multiplier are called factors.

It is proved in Arithmetic (see Davies' Arithmetic, § 22), that the value of a product is not affected by changing the order of its factors: that is,

$$12ab = ab \times 12 = ba \times 12 = a \times 12 \times b.$$

For convenience, however, the letters in each term are generally arranged in alphabetical order, from the left to the right.

Let it be required to multiply $7a^3b^2$ by $4a^2b$.

By decomposing the multiplicand and multiplier into their factors, we may write the product under the form

$$7a^3b^2 \times 4a^2b = 7aaabb \times 4aab;$$

and since we may change the order of the factors without affecting the value of the product, we have,

$$7a^3b^2 \times 4a^2b = 7 \times 4aaaaabbb = 28a^5b^3;$$

a result which is obtained by multiplying the co-efficients together for a new co-efficient, and adding the exponents of the same letter, for the new exponents.

Again: multiply the monomial $12a^2b^4c^2$ by $8a^3b^2d^2$.

We can place the product under the form,

$$12a^2b^4c^2 \times 8a^3b^2d^2 = 12 \times 8aaaaabbbbbbccdd = 96a^5b^6c^2d^2.$$

By considering the manner in which these results are obtained, we see, that any quantity, as a, must be found as many times

a factor in the product, as it is a factor in both the multiplicand and multiplier; which number will always be expressed by the sum of its exponents.

41. Hence, for the multiplication of monomials we have the following

RULE.

I. *Multiply the co-efficients together for a new co-efficient.*

II. *Write after this co-efficient all the letters which enter into the multiplicand and multiplier, affecting each with an exponent equal to the sum of its exponents in both factors.*

EXAMPLES.

(1) - - $8a^2bc^2 \times 7abd^2 = 56a^3b^2c^2d^2$.

(2) - - $21a^3b^2dc \times 8abc^3 = 168a^4b^3c^4d$.

	(3)	(4)	(5)	(6)
Multiply - -	$3a^2b$	$12a^2x$	$6xyz$	a^2xy
by - -	$2ba^2$	$12x^2y$	ay^2z	$2xy^2$
	$6a^4b^2$	$144a^2x^3y$	$6axy^3z^2$	$2a^2x^2y^3$.

7. Multiply $8a^5b^2c$ by $7a^8b^5cd$. *Ans.* $56a^{13}b^7c^2d$.
8. Multiply $5abd^3$ by $12cd^5$. *Ans.* $60abcd^8$.
9. Multiply $7a^4bd^2c^3$ by $abdc$. *Ans.* $7a^5b^2d^3c^4$.

42. We will now proceed to the multiplication of polynomials. In order to explain the most general case, we will suppose the multiplicand and multiplier each to contain additive and subtractive terms.

Let a represent the sum of all the additive terms of the multiplicand, and b the sum of the subtractive terms; c the sum of the additive terms of the multiplier, and d the sum of the subtractive terms. The multiplicand will then be represented by $a - b$ and the multiplier, by $c - d$.

We will now show how the multiplication expressed by $(a - b) \times (c - d)$ can be effected.

$$\begin{array}{r} a - b \\ c - d \\ \hline ac - bc \\ -ad + bd \\ \hline ac - bc - ad + bd. \end{array}$$

The required product is equal to $a - b$ taken as many times as there are units in $c - d$. Let us first multiply by c; that is, take $a - b$ as many times as there are units in c. We begin by writing ac, which is too great by b taken

c times; for it is only the *difference* between a and b, that is first to be multiplied by c. Hence, $ac - bc$ is the product of $a - b$ by c. But the true product is $a - b$ taken $c - d$ times: hence, the last product is too great by $a - b$ taken d times; that is, by $ad - bd$, which must be subtracted. Changing the signs and subtracting this from the first product (Art. 37), we have

$$(a - b) \times (c - d) = ac - bc - ad + bd:$$

If we suppose a and c each equal to 0, the product will reduce to $+ bd$.

43. By considering the product of $a - b$ by $c - d$, we may deduce the following rule for the signs, in the multiplication of two polynomials.

When two terms of the multiplicand and multiplier are affected with the same sign, their product will be affected with the sign $+$, *and when they are affected with contrary signs, their product will be affected with the sign* $-$.

Again, we say in algebraic language, that $+$ multiplied by $+$, or $-$ multiplied by $-$, gives $+$; $-$ multiplied by $+$, or $+$ multiplied by $-$, gives $-$. But since mere signs cannot be multiplied together, this last enunciation does not, in itself, express a distinct idea, and should only be considered as an abbreviation of the preceding.

This is not the only case in which algebraists, for the sake of brevity, employ expressions in a technical sense in order to secure the advantage of fixing the rules in the memory.

44. Hence, for the multiplication of polynomials we have the following

RULE.

Multiply all the terms of the multiplicand by each term of the multiplier in succession, affecting the product of any two terms with the sign plus, when their signs are alike, and with the sign minus, when their signs are unlike. Then reduce the polynomial result to its simplest form.

1. Multiply - - - - - - - $3a^2 + 4ab + b^2$

by - - - - - - - - $2a + 5b$

$$\begin{array}{lrrr} 6a^3 + & 8a^2b + & 2ab^2 & \\ & + 15a^2b + & 20ab^2 + & 5b^3 \end{array}$$

The product after reducing,

becomes - - - - - - $6a^3 + 23a^2b + 22ab^2 + 5b^3$.

(2).

$$\begin{array}{l} x^2 + y^2 \\ x - y \\ \hline x^3 + xy^2 \\ \quad - xy^2 - y^3 \\ \hline x^3 + 0 \quad - y^3 \end{array}$$

(3).

$$\begin{array}{l} x^5 + xy^6 + 7ax \\ ax + 5ax \\ \hline ax^6 + ax^2y^6 + 7a^2x^2 \\ \quad + 5ax^6 + 5ax^2y^6 + 35a^2x^2 \\ \hline 6ax^6 + 6ax^2y^6 + 42a^2x^2. \end{array}$$

4. Multiply $x^2 + 2ax + a^2$ by $x + a$.

Ans. $x^3 + 3ax^2 + 3a^2x + a^3$.

5. Multiply $x^2 + y^2$ by $x + y$.

Ans. $x^3 + xy^2 + x^2y + y^3$.

6. Multiply $3ab^2 + 6a^2c^2$ by $3ab^2 + 3a^2c^2$.

Ans. $9a^2b^4 + 27a^3b^2c^2 + 18a^4c^4$.

7. Multiply $4x^2 - 2y$ by $2y$. *Ans.* $8x^2y - 4y^2$.

8. Multiply $2x + 4y$ by $2x - 4y$. *Ans.* $4x^2 - 16y^2$.

9. Multiply $x^3 + x^2y + xy^2 + y^3$ by $x - y$. *Ans.* ——.

10. Multiply $x^2 + xy + y^2$ by $x^2 - xy + y^2$.

Ans. $x^4 + x^2y^2 + y^4$.

In order to bring together the similar terms, in the product of two polynomials, we arrange the terms of each polynomial with reference to a particular letter.

11. Multiply $4a^3 - 5a^2b - 8ab^2 + 2b^3$

by $2a^2 - 3ab - 4b^2$

$$\begin{array}{l} 8a^5 - 10a^4b - 16a^3b^2 + 4a^2b^3 \\ \quad - 12a^4b + 15a^3b^2 + 24a^2b^3 - 6ab^4 \\ \quad\quad - 16a^3b^2 + 20a^2b^3 + 32ab^4 - 8b^5 \\ \hline 8a^5 - 22a^4b - 17a^3b^2 + 48a^2b^3 + 26ab^4 - 8b^5. \end{array}$$

After having arranged the polynomials, with reference to the letter a, multiply each term of the first, by the term $2a^2$ of the second; this gives the polynomial $8a^5 - 10a^4b - 16a^3b^2 + 4a^2b^3$, the signs of which are the same as those of the multiplicand. Passing then to the term $-3ab$ of the multiplier, multiply each term of the multiplicand by it, and as it is affected with the sign $-$, affect each product with a sign contrary to that of the corresponding term in the multiplicand; this gives

$$-12a^4b + 15a^3b^2 + 24a^2b^3 - 6ab^4$$

The same operation is also performed with the term $-4b^2$, which is also subtractive; this gives,

$$-16a^3b^2+20a^2b^3+32ab^4-8b^5.$$

The product is then reduced, and we finally obtain, for the most simple expression of the product,

$$8a^5-22a^4b-17a^3b^2+48a^2b^3+26ab^4-8b^5.$$

12. Multiply $2a^2-3ax+4x^2$ by $5a^2-6ax-2x^2$.

Ans. $10a^4-27a^3x+34a^2x^2-18ax^3-8x^4$.

13. Multiply $3x^2-2yx+5$ by $x^2+2xy-3$.

Ans. $3x^4+4x^3y-4x^2-4x^2y^2+16xy-15$.

14. Multiply $3x^3+2x^2y^2+3y^2$ by $2x^3-3x^2y^2+5y^3$.

Ans. $\begin{cases} 6x^6-5x^5y^2-6x^4y^4+6x^3y^2+15x^3y^3 \\ -9x^2y^4+10x^2y^5+15y^5. \end{cases}$

15. Multiply $8ax-6ab-c$ by $2ax+ab+c$.

Ans. $16a^2x^2-4a^2bx-6a^2b^2+6acx-7abc-c^2$.

16. Multiply $3a^2-5b^2+3c^2$ by a^2-b^3.

Ans. $3a^4-5a^2b^2+3a^2c^2-3a^2b^3+5b^5-3b^3c^2$.

17. Multiply $3a^2-5bd+cf$

by $-5a^2+4bd-8cf$

Prod. red. $-15a^4+37a^2bd-29a^2cf-20b^2d^2+44bcdf-8c^2f^2$.

18. Multiply $4a^3b^2-5a^2b^2c+8a^2bc^2-3a^2c^3-7abc^3$

by $2ab^2-4abc-2bc^2+c^3$.

Prod. red. $\begin{cases} 8a^4b^4-10a^3b^4c+28a^3b^3c^2-34a^3b^2c^3 \\ -4a^2b^3c^3-16a^4b^3c+12a^3bc^4+7a^2b^2c^4 \\ +14a^2bc^5+14ab^2c^5-3a^2c^6-7abc^6. \end{cases}$

45. *Results deduced from the multiplication of polynomials.*

1st. If the polynomials which are multiplied together are homogeneous,

Their product will also be homogeneous, and the degree of each term will be equal to the sum of the degrees of any two terms of the multiplicand and multiplier.

Thus, in example 18th, each term of the multiplicand is of the 5th degree, and each term of the multiplier of the 3d degree: hence, each term of the product is of the 8th degree. This remark serves to discover any errors in the addition of the exponents.

2d. If no two of the partial products are similar, there will be no reduction among the terms of the entire product: hence,

The total number of terms in the entire product will be equal to the number of terms in the multiplicand multiplied by the number of terms in the multiplier.

This is evident, since *each* term of the multiplier will produce as many terms as there are terms in the multiplicand. Thus, in example 16th, there are three terms in the multiplicand and two in the multiplier: hence, the number of terms in the product is equal to $3 \times 2 = 6$.

3d. Among the different terms of the product, there are always some which cannot be reduced with any others. For, let us consider the product with reference to any letter common to the multiplicand and multiplier. Then, the irreducible terms are,

1st. The term produced by the multiplication of the two terms of the multiplicand and multiplier which contain the highest exponent of this letter; and the term produced by the multiplication of the two terms which contain the lowest exponent of this letter. For, these two partial products will contain this letter, affected with a higher and lower exponent than either of the other partial products, and consequently, they cannot be similar to any of them. This remark, the truth of which is deduced from the law of the exponents, will be very useful in division.

$$\begin{array}{ll}
\text{Multiply - -} & 5a^4b^2 + 3a^2b - ab^4 - 2ab^3 \\
\text{by - - -} & \underline{a^2b \;-\; ab^2 \qquad\qquad\qquad\qquad} \\
\text{Product,} & \left\{\begin{array}{l} 5a^6b^3 + 3a^4b^2 - \;a^3b^5 - 2a^3b^4 \\ \qquad\quad - 5a^5b^4 - 3a^3b^3 + \;a^2b^6 + 2a^2b^5. \end{array}\right.
\end{array}$$

If we examine the multiplicand and multiplier, with reference to a, we see that the product of $5a^4b^2$ by a^2b, must be irreducible; also, the product of $-2ab^3$ by ab^2. If we consider the letter b, we see that the product of $-ab^4$ by $-ab^2$, must be irreducible, also that of $3a^2b$ by a^2b.

46. We will apply the rules for the multiplication of algebraic quantities in the demonstration of the following theorems.

THEOREM I.

The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.

Let a denote one of the quantities and b the other: then

$$a + b = \text{ their sum.}$$

Now, we have from known principles,

$$(a + b)^2 = (a + b) \times (a + b) = a^2 + 2ab + b^2,$$

which result is the enunciation of the theorem in the language of Algebra.

To apply this result to finding the square of the binomial

$$5a^2 + 8a^2b,$$

we have $(5a^2 + 8a^2b)^2 = 25a^4 + 80a^4b + 64a^4b^2$.

Also, $(6a^4b + 9ab^3) = 36a^8b^2 + 108a^5b^4 + 81a^2b^6$;

also, $(8a^3 + 7acb)^2 =$.

THEOREM II.

The square of the difference between two quantities is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second.

Let a represent one of the quantities and b the other: then

$$a - b = \text{ their difference.}$$

Now, we have from known principles,

$$(a - b)^2 = (a - b) \times (a - b) = a^2 - 2ab + b^2,$$

which is the algebraic enunciation of the theorem.

To apply this to an example, we have

$$(7a^2b^2 - 12ab^3) = 49a^4b^4 - 168a^3b^5 + 144a^2b^6.$$

Also, $(4a^3b^3 - 7c^2d^3)^2 =$

THEOREM III.

The product of the sum of two quantities multiplied by their difference, is equal to the difference of their squares.

Let the quantities be denoted by a and b.

Then, $a + b =$ their sum, and $a - b =$ their difference.

We have, from known principles,

$$(a + b) \times (a - b) = a^2 - b^2,$$

which is the algebraic enunciation of the theorem.

To apply this principle to an example, we have

$$(8a^3 + 7ab^2) \times (8a^3 - 7ab^2) = 64a^6 - 49a^2b^4.$$

Also, $(9a^6c + 7ab^5) \times (9a^6c - 7ab^5) =$

47. By considering the last three results, it will be perceived that their composition, or the manner in which they are formed from the multiplicand and multiplier, is entirely independent of any particular values that may be attributed to the letters a and b which enter the two factors.

The manner in which an algebraic product is formed from its two factors, is called the *law* of this product; and this law remains always the same, whatever values may be attributed to the letters which enter into the two factors.

Of factoring Polynomials.

48. A given polynomial may often be resolved into two factors by mere inspection. This is generally done by selecting all the factors common to every term of the polynomial for one factor, and writing what remains of each term within a parenthesis for the other factor.

1. Take, for example, the polynomial

$$ab + ac;$$

in which, it is plain, that a is a factor of both terms: hence

$$ab + ac = a(b + c).$$

2. Take, for a second example, the polynomial

$$ab^2c + 5ab^3 + ab^2c^2.$$

It is plain that a and b^2 are factors of all the terms: hence

$$ab^2c + 5ab^3 + ab^2c^2 = ab^2(c + 5b + c^2),$$

3. Take the polynomial $25a^4 - 30a^3b + 15a^2b^2$; it is evident that 5 and a^2 are factors of each of the terms. We may, therefore, put the polynomial under the form

$$5a^2(5a^2 - 6ab + 3b^2).$$

4. Find the factors of $3a^2b + 9a^2c + 18a^2xy$.

Ans. $3a^2(b + 3c + 6xy)$.

5. Find the factors of $8a^2cx - 18acx^2 + 2ac^5y - 30a^6c^9x$.

Ans. $2ac(4ax - 9x^2 + c^4y - 15a^5c^8x)$.

6. Find the factors of $24a^2b^2cx - 30a^8b^5c^6y + 36a^7b^8cd + 6abc$.

Ans. $6abc(4abx - 5a^7b^4c^5y + 6a^6b^7d + 1)$.

7. Find the factors of $a^2 + 2ab + b^2$.

Ans. $(a + b) \times (a + b)$.

8. Find the factors of $a^2 - b^2$. *Ans.* $(a + b) \times (a - b)$.

9. Find the factors of $a^2 - 2ab + b^2$.

Ans. $(a - b) \times (a - b)$.

10. Find the factors of the polynomial $6a^3b + 8a^2b^5 - 16ab^7 - 2ab$.

11. Find the factors of the polynomial $15abc^2 - 3bc^2 + 9a^3b^5c^6 - 12db^6c^2$.

12. Find the factors of the polynomial $25a^6bc^6 - 30a^8bc^4d - 5ac^4 - 60ac^6$.

13. Find the factors of the polynomial $42a^2b^2 - 7abcd + 7abd$.

Ans. $7ab\,(6ab - cd + d)$.

DIVISION.

49. Division, in Algebra, explains the method of finding from two given quantities, a third quantity, which multiplied by the first shall produce the second.

The first of the given quantities is called the *divisor:* the second, the *dividend;* and the third, or quantity sought, the *quotient.*

Let us first consider the case of two monomials, and divide $35a^3b^2c$ by $7ab$.

The division may be indicated thus,

$$\frac{35a^3b^2c^2}{7ab} = 5a^{3-1}b^{2-1}c^2 = 5a^2bc^2.$$

Now, since the quotient must be such a quantity as multiplied by the divisor will produce the dividend, the co-efficient of the quotient multiplied by 7 must give 35, the co-efficient of the dividend; hence, the new co-efficient 5 is found by dividing 35 by 7. Again, the exponent of any letter, as a, in the quotient, added to the exponent of the same letter in the divisor, must give the exponent of this letter in the dividend: hence, the exponent in the quotient is found by subtracting the exponent in the divisor from that in the dividend. Thus, the exponent of

$$a \text{ is } 3 - 1 = 2, \text{ and of } b,\ 2 - 1 = 1,$$

and since c is not found in the divisor, there is nothing to be subtracted from its exponent.

50. Hence, for the division of monomials, we have the following

RULE.

I. *Divide the co-efficient of the dividend by the co-efficient of the divisor, for a new co-efficient.*

II. *Write after this co-efficient, all the letters of the dividend, and affect each with an exponent equal to the excess of its exponent in the dividend over that in the divisor.*

From this rule we find,

$$\frac{48a^3b^3c^2d}{12ab^2c} = 4a^2bcd; \qquad \frac{150a^5b^3cd^3}{30a^3d^2} = 5a^2b^3cd.$$

1. Divide $16x^2$ by $8x$. *Ans.* $2x$
2. Divide $15a^2xy^3$ by $3ay$. *Ans.* $5axy^2$.
3. Divide $84ab^3x$ by $12b^2$. *Ans.* $7abx$.
4. Divide $96a^4b^2c^3$ by $12a^2bc$. *Ans.* $8a^2bc^2$.
5. Divide $144a^9b^8c^7d^5$ by $36a^4b^6c^6d$. *Ans.* $4a^5b^2cd^4$.
6. Divide $256a^3bc^2x^3$ by $16a^2cx^2$. *Ans.* $16abcx$.
7. Divide $300a^5b^4c^3x^2$ by $30a^4b^3c^2x$. *Ans.* $10abcx$.

51. It follows from the preceding rule that the exact division of monomials will be impossible.

1st. When the co-efficient of the dividend is not divisible by that of the divisor.

2d. When the exponent of the same letter is greater in the divisor than in the dividend.

3d. When the divisor contains one or more letters which are not found in the dividend.

When either of these three cases occurs, the quotient remains under the form of a monomial fraction; that is, a monomial expression, necessarily containing the algebraic sign of division. Such expressions may frequently be reduced.

Take, for example, $\dfrac{12a^4b^2cd}{8a^2bc^2} = \dfrac{3a^2bd}{2c}$

Here an entire monomial cannot be obtained for a quotient; for, 12 is not divisible by 8, and moreover, the exponent of c is less in the dividend than in the divisor. But the expression can be reduced, by dividing the numerator and denominator by the factors 4, a^2, b, and c, which are common to both the terms of the fraction.

In general, to reduce a monomial fraction, we have the following

RULE.

Suppress all the factors common to the numerator and denominator, and write those letters which are not common, with their respective exponents, in the term of the fraction which contains them.

From this rule we find,

$$\frac{48a^3b^2cd^3}{36a^2b^3c^2de} = \frac{4ad^2}{3bce} \quad \text{and} \quad \frac{37ab^3c^5d}{6a^3bc^4d^2} = \frac{37b^2c}{6a^2d};$$

also, $$\frac{12a^8b^6c^7}{16a^7b^5c^9} = \frac{3ab}{4c^2}; \quad \text{and} \quad \frac{7a^2b}{14a^3b^2} = \frac{1}{2ab};$$

In the last example, as all the factors of the dividend are found in the divisor, the numerator is reduced to *unity;* for, in fact, both terms of the fraction are divided by the numerator.

52. It often happens, that the exponents of certain letters, are the same in the dividend and divisor.

For example, - - $\frac{24a^3b^2}{8a^2b^2} = 3a,$

is a case in which the letter b is affected with the same exponent in the dividend and divisor: hence, it will divide out, and will not appear in the quotient.

But if it is desirable to preserve the trace of this letter in the quotient, we may apply to it the rule for the exponents (Art. 50): which gives

$$\frac{b^2}{b^2} = b^{2-2} = b^0$$

This new symbol b^0, indicates that the letter b enters 0 times as a factor in the quotient (Art. 13); or what is the same thing, that it does not enter it at all.. Still, the notation shows that b was in the dividend and divisor with the same exponent, and has disappeared by division.

In like manner, $\frac{15a^2b^3c^2}{3a^2bc^2} = 5a^0b^2c^0 = 5b^2.$

53. We will now show that the power of any quantity whose exponent is 0, is equal to unity. Let the quantity be represented by a, and let m denote any exponent whatever.

Then, $\frac{a^m}{a^m} = a^{m-m} = a^0$, by the rule of division.

But, $\frac{a^m}{a^m} = 1$, since the numerator and denominator are equal:

hence, $a^0 = 1$, since each is equal to $\frac{a^m}{a^m}$.

We observe again, that the symbol a^0 is only employed conventionally, to preserve in the calculation the trace of a letter which entered in the enunciation of a question, but which may disappear by division.

Division of Polynomials.

54. The object of division, is to find a third polynomial called the quotient, which, multiplied by the divisor, shall produce the dividend.

Hence, the dividend is the assemblage, after reduction, of the partial products of each term of the divisor by each term of the quotient, and consequently, the signs of the terms in the quotient must be such as to give proper signs to the partial products.

Since, in multiplication, the product of two terms having the same sign is affected with the sign $+$, and the product of two terms having contrary signs, with the sign $-$, we may conclude,

1st. That when the term of the dividend has the sign $+$, and that of the divisor the sign of $+$, the term of the quotient must have the sign $+$.

2d. When the term of the dividend has the sign $+$, and that of the divisor the sign $-$, the term of the quotient must have the sign $-$; because it is only the sign $-$, which, combined with the sign $-$, can produce the sign $+$ of the dividend.

3d. When the term of the dividend has the sign $-$, and that of the divisor the sign $+$, the quotient must have the sign $-$.

That is, when the two corresponding terms of the dividend and divisor have the same sign, their quotient will be affected with the sign $+$, and when they are affected with contrary signs, their quotient will be affected with the sign $-$; again, for the sake of brevity, we say that

$+$ divided by $+$, and $-$ divided by $-$, give $+$;
$-$ divided by $+$, and $+$ divided by $-$, give $-$.

FIRST EXAMPLE.

Divide $a^2 - 2ax + x^2$ by $a - x$.

It is found most convenient, in division in algebra, to place the divisor on the right of the dividend and the quotient directly under the divisor.

$$\begin{array}{r|l} \textit{Dividend.} & \textit{Divisor.} \\ a^2 - 2ax + x^2 & \underline{a - x} \\ \underline{a^2 - ax } & a - x \\ - ax + x^2 & \textit{Quotient.} \\ \underline{- ax + x^2}. & \end{array}$$

We first divide the term a^2 of the dividend by the term a of the divisor, the partial quotient is a, which we place under the divisor. We then multiply the divisor by a, and subtract the product $a^2 - ax$ from the dividend, and to the remainder bring down x^2. We then divide the first term of the remainder, $- ax$, by a, the quotient is $- x$. We then multiply the divisor by $- x$, and, subtracting as before, we find nothing remains. Hence, $a - x$ is the exact quotient.

SECOND EXAMPLE.

Let it be required to divide $26a^2b^2 + 10a^4 - 48a^3b + 24ab^3$ by $4ab - 5a^2 + 3b^2$. In order that we may follow the steps of the operation more easily, we will arrange the quantities with reference to the letter a.

$$\begin{array}{r|l} \textit{Dividend.} & \textit{Divisor.} \\ 10a^4 - 48a^3b + 26a^2b^2 + 24ab^3 & \underline{- 5a^2 + 4ab + 3b^2} \\ \underline{+ 10a^4 - 8a^3b - 6a^2b^2 } & - 2a^2 + 8ab \\ - 40a^3b + 32a^2b^2 + 24ab^3 & \textit{Quotient.} \\ \underline{- 40a^3b + 32a^2b^2 + 24ab^3}. & \end{array}$$

It follows from the definition of division, and the rule for the multiplication of polynomials (Art. 44), that the dividend is the assemblage, after addition and reduction, of the partial products of each term of the divisor, by each term of the quotient sought. Hence, if we could discover a term in the dividend which was derived, without reduction, from the multiplication of a term of the divisor by a term of the quotient, then, by dividing this term of the dividend by that of the divisor, we should obtain a term of the required quotient.

Now, from the third remark of Art. 45, the term $10a^4$, affected with the highest exponent of the letter a, is derived, without reduction from the two terms of the divisor and quotient, affected

with the highest exponent of the same letter. Hence, by dividing the term $10a^4$ by the term $-5a^2$, we shall have a term of the required quotient.

$$\begin{array}{l|l}
\textit{Dividend.} & \textit{Divisor.} \\
10a^4 - 48a^3b + 26a^2b^2 + 24ab^3 & -5a^2 + 4ab + 3b^2 \\
\hline
+10a^4 - 8a^3b - 6a^2b^2 & -2a^2 + 8ab \\
\quad\; -40a^3b + 32a^2b^2 + 24ab^3 & \textit{Quotient.} \\
\quad\; -40a^3b + 32a^2b^2 + 24ab^3. &
\end{array}$$

Since the terms $10a^4$ and $-5a^2$ are affected with contrary signs, their quotient will have the sign $-$; hence, $10a^4$, divided by $-5a^2$, gives $-2a^2$ for a term of the required quotient.

After having written this term under the divisor, multiply each term of the divisor by it, and subtract the product,

$$10a^4 - 8a^3b + 6a^2b^2,$$

from the dividend, which is done by writing it below the dividend, conceiving the signs to be changed, and performing the reduction. Thus, the remainder after the first partial division is

$$-40a^3b + 32a^2b^2 + 24ab^3.$$

This result is composed of the partial products of each term of the divisor, by all the terms of the quotient which remain to be determined. We may then consider it as a new dividend, and reason upon it as upon the proposed dividend. We will therefore divide the term $-40a^3b$, affected with the highest exponent of a, by the term $-5a^2$ of the divisor. Now, from the preceding principles,

$$-40a^3b, \text{ divided by } -5a^2, \text{ gives } +8ab$$

for a new term of the quotient, which is written on the right of the first. Multiplying each term of the divisor by this term of the quotient, and writing the products underneath the second dividend, and making the subtraction, we find that nothing remains. Hence,

$$-2a^2 + 8ab \quad \text{or} \quad 8ab - 2a^2$$

is the required quotient, and if the divisor be multiplied by it, the product will be the given dividend.

By considering the preceding reasoning, we see that, in each partial operation, we divide that term of the dividend which is

affected with the highest exponent of one of the letters, by that term of the divisor affected with the highest exponent of the same letter. Now, we avoid the trouble of looking out these terms *by writing, in the first place, the terms of the dividend and divisor in such a manner that the exponents of the same letter shall go on diminishing from left to right.*

This is what is called *arranging* the dividend and divisor with reference to a certain letter. By this preparation, the first term on the left of the dividend, and the first on the left of the divisor, are always the two which must be divided by each other in order to obtain a term of the quotient.

55. Hence, for the division of polynomials we have the following

RULE.

I. *Arrange the dividend and divisor with reference to a certain letter, and then divide the first term on the left of the dividend by the first term on the left of the divisor, for the first term of the quotient; muliply the divisor by this term and subtract the product from the dividend.*

II. *Then divide the first term of the remainder by the first term of the divisor, for the second term of the quotient; multiply the divisor by this second term, and subtract the product from the result of the first operation. Continue the same process, and if the remainder is* 0, *the division is said to be exact.*

THIRD EXAMPLE.

Divide $21x^3y^2 + 25x^2y^3 + 68xy^4 - 40y^5 - 56x^5 - 18x^4y$ by $5y^2 - 8x^2 - 6xy$.

$$
\begin{array}{lll}
 & -40y^5 + 68xy^4 + 25x^2y^3 + 21x^3y^2 - 18x^4y - 56x^5 & \| \; 5y^2 - 6xy - 8x^2 \\
 & -40y^5 + 48xy^4 + 64x^2y^3 & \quad -8y^3 + 4xy^2 - 3x^2y + 7x^3 \\
\text{1st rem.} & 20xy^4 - 39x^2y^3 + 21x^3y^2 & \\
 & 20xy^4 - 24x^2y^3 - 32x^3y^2 & \\
\text{2d rem.} & \quad -15x^2y^3 + 53x^3y^2 - 18x^4y & \\
 & \quad -15x^2y^3 + 18x^3y^2 + 24x^4y & \\
\text{3d rem.} & \qquad 35x^3y^2 - 42x^4y - 56x^5 & \\
 & \qquad 35x^3y^2 - 42x^4y - 56x^5 & \\
\text{Final remainder} & \qquad\quad 0. &
\end{array}
$$

56. REMARK.—In performing the division, it is not necessary to bring down all the terms of the dividend to form the first remainder, but they may be brought down in succession, as in the example.

As it is important that beginners should render themselves familiar with the algebraic operation, and acquire the habit of calculating promptly, we will treat this last example in a different manner, at the same time indicating the simplifications which should be introduced. These, consist in subtracting each partial product from the dividend as soon as this product is formed.

$$\begin{array}{l|l}
-40y^5+68xy^4+25x^2y^3+21x^3y^2-18x^4y-56x^5 & 5y^2-6xy-8x^2 \\
\hline
\text{1st rem.}\quad 20xy^4-39x^2y^3+21x^3y^2 & -8y^3+4xy^2-3x^2y+7x^3 \\
\text{2d rem.}\quad -15x^2y^3+53x^3y^2-18x^4y & \\
\text{3d rem.}\quad -35x^3y^2-42x^4y-56x^5 & \\
\text{Final remainder}\quad 0. &
\end{array}$$

First, by dividing $-40y^5$ by $5y^2$, we obtain $-8y^3$ for the quotient. Multiplying $5y^2$ by $-8y^3$, we have $-40y^5$, or by changing the sign, $+40y^5$, which destroys the first term of the dividend.

In like manner, $-6xy \times -8y^3$ gives $+48xy^4$, and for the subtraction $-48xy^4$, which reduced with $+68xy^4$, gives $20xy^4$ for a remainder. Again, $-8x^2 \times -8y^3$ gives $+$, and changing the sign, $-64x^2y^3$, which reduced with $25x^2y^3$, gives $-39x^2y^3$ Hence, the result of the first operation is $20xy^4-39x^2y^3$, followed by those terms of the dividend which have not been reduced with the partial products already obtained. For the second part of the operation, it is only necessary to bring down the next term of the dividend, to separate this new dividend from the primitive by a line, and to operate upon this new dividend in the same manner as we operated upon the primitive, and so on.

FOURTH EXAMPLE.

Divide - - $95a-73a^2+56a^4-25-59a^3$ by $-3a^2+5-11a-7a^3$.

$$\begin{array}{l|l}
56a^4-59a^3-73a^2+95a-25 & 7a^3-3a^2-11a+5 \\
\hline
\text{1st rem.}\quad -35a^3+15a^2+55a-25 & 8a\ -5 \\
\text{2d remainder}\quad 0. &
\end{array}$$

GENERAL EXAMPLES.

1. Divide $10ab + 15ac$ by $5a$. *Ans.* $2b + 3c$.

2. Divide $30ax - 54x$ by $6x$. *Ans.* $5a - 9$.

3. Divide $10x^2y - 15y^2 - 5y$ by $5y$. *Ans.* $2x^2 - 3y - 1$.

4. Divide $12a + 3ax - 18ax^2$ by $3a$. *Ans.* $4 + x - 6x^2$.

5. Divide $6ax^2 + 9a^2x + a^2x^2$ by ax. *Ans.* $6x + 9a + ax$.

6. Divide $a^2 + 2ax + x^2$ by $a + x$. *Ans.* $a + x$.

7. Divide $a^3 - 3a^2y + 3ay^2 - y^3$ by $a - y$.
Ans. $a^2 - 2ay + y^2$.

8. Divide $24a^2b - 12a^3cb^2 - 6ab$ by $-6ab$.
Ans. $-4a + 2a^2cb + 1$.

9. Divide $6x^4 - 96$ by $3x - 6$. *Ans.* $2x^3 + 4x^2 + 8x + 16$.

10. Divide - - $a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$ by $a^2 - 2ax + x^2$. *Ans.* $a^3 - 3a^2x + 3ax^2 - x^3$.

11. Divide $48x^3 - 76ax^2 - 64a^2x + 105a^3$ by $2x - 3a$.
Ans. $24x^2 - 2ax - 35a^2$.

12. Divide $y^6 - 3y^4x^2 + 3y^2x^4 - x^6$ by $y^3 - 3y^2x + 3yx^2 - x^3$.
Ans. $y^3 + 3y^2x + 3yx^2 + x^3$.

13. Divide $64a^4b^6 - 25a^2b^8$ by $8a^2b^3 + 5ab^4$.
Ans. $8a^2b^3 - 5ab^4$.

14. Divide $6a^3 + 23a^2b + 22ab^2 + 5b^3$ by $3a^2 + 4ab + b^2$.
Ans. $2a + 5b$.

15. Divide $6ax^6 + 6ax^2y^6 + 42a^2x^2$ by $ax + 5ax$.
Ans. $x^5 + xy^6 + 7ax$.

16. Divide $-15a^4 + 37a^2bd - 29a^2cf - 20b^2d^2 + 44bcdf - 8c^2f^2$ by $3a^2 - 5bd + cf$. *Ans.* $-5a^2 + 4bd - 8cf$.

17. Divide $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.
Ans. $x^2 + xy + y^2$.

18. Divide $x^4 - y^4$ by $x - y$. *Ans.* $x^3 + x^2y + xy^2 + y^3$.

19. Divide $3a^4 - 8a^2b^2 + 3a^2c^2 + 5b^4 - 3b^2c^2$ by $a^2 - b^2$.
Ans. $3a^2 - 5b^2 + 3c^2$.

20. Divide $6x^6 - 5x^5y^2 - 6x^4y^4 + 6x^3y^2 + 15x^3y^3 - 9x^2y^4 + 10x^2y^5 + 15y^5$ by $3x^3 + 2x^2y^2 + 3y^2$. *Ans.* $2x^3 - 3x^2y^2 + 5y^3$

Remarks on the Division of Polynomials.

57. When the first term of the arranged dividend is not exactly divisible by that of the arranged divisor, the complete division is impossible; that is to say, there is not a polynomial which, multiplied by the divisor, will produce the dividend. And in general, we shall find that a division is impossible, *when the first term of any one of the partial dividends is not divisible by the first term of the divisor.*

We will add, as to polynomials, that it may often be discovered by mere inspection that they are not divisible. When the polynomials contain two or more letters, observe the two terms of the dividend and divisor, which are affected with the highest exponent of each of the letters. If these terms do not give an exact quotient, we may conclude that the total division is impossible.

Take, for example,

$$12a^3 - 5a^2b + 7ab^2 - 11b^3 \,\|\, \underline{4a^2 + 8ab + 3b^2}$$

By considering only the letter a, the division would appear possible; but regarding the letter b, the division is impossible, since $-11b^3$ is not divisible by $3b^2$.

58. One polynomial A, cannot be divided by another B containing a letter which is not found in the dividend; for, it is impossible that a third quantity, multiplied by B which contains a certain letter, should give a product independent of that letter.

A monomial is never divisible by a polynomial, because every polynomial multiplied by either a monomial or a polynomial gives a product containing at least two terms which are not susceptible of reduction.

59. If the letter, with reference to which the dividend is arranged, is not found in the divisor, *the divisor is said to be independent of that letter;* and in that case, the exact division is impossible, *unless the divisor will divide separately the co-efficient of each term of the dividend.*

For example, if the dividend were

$$3ba^4 + 9ba^2 + 12b,$$

arranged with reference to the letter a, and the divisor $3b$, the

divisor would be *independent* of the letter a; and it is evident that the exact division could not be performed unless the co-efficient of each term of the dividend were divisible by $3b$. The exponents of the leading letter in the quotient would be the same as in the dividend.

1. Divide $18a^3x^2 - 36a^2x^3 - 12ax$ by $6x$.

Ans. $3a^3x - 6a^2x^2 - 2a$.

2. Divide $25a^4b - 30a^2b + 40ab$ by $5b$.

Ans. $5a^4 - 6a^2 + 8a$.

60. Although there is some analogy between arithmetical and algebraical division, with respect to the manner in which the operations are disposed and performed, yet there is this essential difference between them, that in arithmetical division the figures of the quotient are obtained by trial, while in algebraical division the quotient obtained by dividing the first term of the partial dividend by the first term of the divisor, is always one of the terms of the quotient sought.

From the third remark of Art. 45, it appears that the term of the dividend affected with the highest exponent of the leading letter, and the term affected with the lowest exponent of the same letter, may each be derived without reduction, from the multiplication of a term of the divisor by a term of the quotient. Therefore, nothing prevents our commencing the operation at the right instead of the left, since it might be performed upon the terms affected with the lowest exponent of the letter, with reference to which the arrangement has been made.

Lastly, so independent are the partial operations required by the process, that after having subtracted the product of the divisor by the first term found in the quotient, we could obtain another term of the quotient by dividing by each other the two terms of the new dividend and divisor, affected with the highest exponent of a different letter from the one first selected. If the same letter is preserved, it is only because there is no reason for changing it, and because the two polynomials are already arranged with reference to it; the first terms on the left of the dividend and divisor being sufficient to obtain a term of the quotient; whereas, if the letter is changed, it would be necessary to seek again for the highest exponent of this letter.

61. Among the different examples of algebraic division, there is one remarkable for its applications. It is expressed thus:

The difference between the same powers of any two quantities is always divisible by the difference between the quantities.

Let the quantities be represented by a and b; and let m denote any positive whole number. Then,

$$a^m - b^m$$

will express the difference between the same powers of a and b, and it is to be proved that $a^m - b^m$ is exactly divisible by $a - b$.

If we begin the division of $a^m - b^m$ by $a - b$, we have

$$\begin{array}{l|l} a^m - b^m & a - b \\ \hline a^m - a^{m-1}b & a^{m-1} \\ \hline \end{array}$$

1st rem. - - - - - $a^{m-1}b - b^m$

or, by factoring - - - $b\,(a^{m-1} - b^{m-1})$.

Dividing a^m by a the quotient is a^{m-1}, by the rule for the exponents. The product of $a - b$ by a^{m-1} being subtracted from the dividend, the first remainder is $a^{m-1}b - b^m$, which can be put under the form $b\,(a^{m-1} - b^{m-1})$.

Now, if the factor $(a^{m-1} - b^{m-1})$ of the remainder, be divisible by $a - b$, it follows that the dividend $a^m - b^m$ is also divisible by $a - b$: that is,

If the difference of the same powers of two quantities be divisible by the difference of the quantities, then, the difference of the powers of a degree greater by unity is also divisible by it.

But by the rules for division, we have

$$\frac{a^2 - b^2}{a - b} = a + b.$$

Hence, we know, from what has just been proved, that $a^3 - b^3$ is divisible by $a - b$, and from that result we conclude that $a^4 - b^4$ is divisible by $a - b$, and so on, until we reach any exponent at m.

CHAPTER III.

OF ALGEBRAIC FRACTIONS.

62. ALGEBRAIC fractions are to be considered in the same point of view as arithmetical fractions; that is, *a unit is supposed to be divided into as many equal parts as there are units in the denominator, and one of these parts is supposed to be taken as many times as there are units in the numerator.*

Thus, in the fractional expression

$$\frac{a+b}{c+d},$$

a given unit is supposed to be divided into as many equal parts as there are units in $c+d$, and as many of these parts are taken, as there are units in $a+b$.

The rules for performing Addition, Subtraction, Multiplication, and Division, are the same as in arithmetical fractions. Hence, it will not be necessary to demonstrate these rules, and in their application we must follow the methods already indicated in similar operations on entire algebraic quantities.

63. Every quantity which is not expressed under a fractional form, is called an *entire* algebraic quantity.

64. An algebraic expression, composed partly of an entire quantity and partly of a fraction, is called a *mixed quantity*.

65. When the division of two monomial quantities cannot be performed exactly, it is indicated by means of the known sign, and in this case, the quotient is presented under the form of a fraction, which we have already learned how to simplify (Art. 51).

With respect to polynomial fractions, the following are cases which are easily reduced.

Take, for example, the expression $\dfrac{a^2 - b^2}{a^2 - 2ab + b^2}$.

This fraction can take the form $\dfrac{(a+b)(a-b)}{(a-b)(a-b)}$ (Art. 46).

Suppressing the factor $a - b$, which is common to the two terms, we obtain $\dfrac{a+b}{a-b}$.

Again, take the expression $\dfrac{5a^3 - 10a^2b + 5ab^2}{8a^3 - 8a^2b}$;

which can be put under the form (Art. 48):

$$\frac{5a(a^2 - 2ab + b^2)}{8a^2(a-b)};$$

which is equal to $\dfrac{5a(a-b)^2}{8a^2(a-b)}$;

and by suppressing the common factors, $a(a-b)$, the result is

$$\frac{5(a-b)}{8a};$$

In the particular cases examined above, the two terms of the fraction are decomposed into factors, and then the factors common to the numerator and denominator are cancelled. Practice teaches the manner of performing these decompositions, when they are possible.

But the two terms of the fraction may be complicated polynomials, and then, their decomposition into factors not being so easy, we have recourse to the process for finding *the greatest common divisor*, which is explained at page 300.

CASE I.

70. To reduce a fraction to its simplest form.

RULE.

I. *Decompose the numerator and denominator into factors, as in Art.* 48.

II. *Then cancel the factors common to the numerator and denominator, and the result will be the simplest form of the fraction.*

EXAMPLES.

1. Reduce the fraction $\frac{3ab + 6ac}{3ad + 12a}$ to its simplest form.

We see, by inspection, that 3 and a are factors of the numerator, hence

$$3ab + 6ac = 3a(b + 2c)$$

We also see, that 3 and a are factors of the denominator, hence

$$3ad + 12a = 3a(d + 4)$$

Hence, $$\frac{3ab + 6ac}{3ad + 12a} = \frac{3a(b + 2c)}{3a(d + 4)} = \frac{b + 2c}{d + 4}.$$

2. Reduce $\frac{6a^2b + 3ac}{9ab + 3ad}$ to its simplest form.

Ans. $\frac{2ab + c}{3b + d}$.

3. Reduce $\frac{25bc + 5bf}{35b^2 + 15b}$ to its lowest terms.

Ans. $\frac{5c + f}{7b + 3}$.

4. Reduce $\frac{54abc}{45a^2c + 9acd}$ to its simplest form.

Ans. $\frac{6b}{5a + d}$.

5. Reduce $\frac{36a^2b + 12abf}{84ab^2}$ to its simplest form.

Ans. $\frac{3a + f}{7b}$

6. Reduce $\frac{12acd - 4cd^2}{12cdf + 4c^2d}$ to its simplest form.

Ans. $\frac{3a - d}{3f + c}$.

7. Reduce $\frac{18a^2c^2 - 3acf}{27ac^2 - 6ac^3}$ to its simplest form.

Ans. $\frac{6ac - f}{9c - 2c^2}$.

CASE II.

71. To reduce a mixed quantity to the form of a fraction.

RULE.

Multiply the entire part by the denominator of the fraction: then connect this product with the terms of the numerator by the rules for addition, and under the result place the given denominator.

EXAMPLES.

1. Reduce $x - \dfrac{(a^2 - x^2)}{x}$ to the form of a fraction.

$$Ans.\ x - \frac{a^2 - x^2}{x} = \frac{x^2 - (a^2 - x^2)}{x} = \frac{2x^2 - a^2}{x}.$$

2. Reduce $x - \dfrac{ax + x^2}{2a}$ to the form of a fraction.

$$Ans.\ \frac{ax - x^2}{2a}.$$

3. Reduce $5 + \dfrac{2x - 7}{3x}$ to the form of a fraction.

$$Ans.\ \frac{17x - 7}{3x}.$$

4. Reduce $1 - \dfrac{x - a - 1}{a}$ to the form of a fraction.

$$Ans.\ \frac{2a - x + 1}{a}.$$

5. Reduce $1 + 2x - \dfrac{x - 3}{5x}$ to the form of a fraction.

$$Ans.\ \frac{10x^2 + 4x + 3}{5x}.$$

6. Reduce $3x - 1 - \dfrac{x + a}{3a - 2}$ to the form of a fraction.

$$Ans.\ \frac{9ax - 4a - 7x + 2}{3a - 2}.$$

CASE III.

72. To reduce a fraction to an entire or mixed quantity.

RULE.

Divide the numerator by the denominator for the entire part, and place the remainder, if any, over the denominator for the fractional part.

EXAMPLES.

1. Reduce $\frac{ax + a^2}{x}$ to a mixed quantity.

$$Ans.\ \frac{ax + a^2}{x} = a + \frac{a^2}{x}.$$

2. Reduce $\frac{ax - x^2}{x}$ to an entire or mixed quantity.

$$Ans.\ a - x.$$

3. Reduce $\frac{ab - 2a^2}{b}$ to a mixed quantity.

$$Ans.\ a - \frac{2a^2}{b}.$$

4. Reduce $\frac{a^2 - x^2}{a - x}$ to an entire quantity.

$$Ans.\ a + x.$$

5. Reduce $\frac{x^3 - y^3}{x - y}$ to an entire quantity.

$$Ans.\ x^2 + xy + y^2.$$

6. Reduce $\frac{10x^2 - 5x + 3}{5x}$ to a mixed quantity.

$$Ans.\ 2x - 1 + \frac{3}{5x}.$$

CASE IV.

73. To reduce fractions having different denominators to equivalent fractions having a common denominator.

RULE.

Multiply each numerator into all the denominators except its own, for the new numerators, and all the denominators together for a common denominator.

EXAMPLES.

1. Reduce $\frac{a}{b}$ and $\frac{b}{c}$ to equivalent fractions having a common denominator.

$$\left.\begin{array}{l} a \times c = ac \\ b \times b = b^2 \end{array}\right\} \text{the new numerators.}$$

and $b \times c = bc$ the common denominator.

2. Reduce $\frac{a}{b}$ and $\frac{a + b}{c}$ to fractions having a common denominator.

$$Ans.\ \frac{ac}{bc} \text{ and } \frac{ab + b^2}{bc}.$$

3. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$, and d, to fractions having a common denominator. *Ans.* $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$, and $\frac{6acd}{6ac}$.

4. Reduce $\frac{3}{4}$, $\frac{2x}{3}$, and $a + \frac{2x}{a}$, to fractions having a common denominator. *Ans.* $\frac{9a}{12a}$, $\frac{8ax}{12a}$, and $\frac{12a^2 + 24x}{12a}$.

5. Reduce $\frac{1}{2}$, $\frac{a^2}{3}$, and $\frac{a^2 + x^2}{a + x}$, to fractions having a common denominator.

Ans. $\frac{3a + 3x}{6a + 6x}$, $\frac{2a^3 + 2a^2x}{6a + 6x}$, and $\frac{6a^2 + 6x^2}{6a + 6x}$.

6. Reduce $\frac{a}{a - b}$, $\frac{c - b}{ax}$, and $\frac{b}{c}$, to fractions having a common denominator.

Ans. $\frac{a^2cx}{a^2cx - abcx}$, $\frac{ac^2 - abc - bc^2 + cb^2}{a^2cx - abcx}$, $\frac{+ a^2bx - ab^2x}{a^2cx - abcx}$.

CASE V.

74. To add fractional quantities together.

RULE.

Reduce the fractions, if necessary, to a common denominator: then add the numerators together and place their sum over the common denominator.

EXAMPLES.

1. Find the sum of $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$.

Here, $\left.\begin{array}{l} a \times d \times f = adf \\ c \times b \times f = cbf \\ e \times b \times d = ebd \end{array}\right\}$ the new numerators.

And $b \times d \times f = bdf$ the common denominator.

Hence, $\frac{adf}{bdf} + \frac{cbf}{bdf} + \frac{ebd}{bdf} = \frac{adf + cbf + ebd}{bdf}$ the sum.

2. To $a - \frac{3x^2}{b}$ add $b + \frac{2ax}{c}$.

Ans. $a + b + \frac{2abx - 3cx^2}{bc}$.

3. Add $\frac{x}{2}$, $\frac{x}{3}$, and $\frac{x}{4}$ together. *Ans.* $x + \frac{x}{12}$.

4. Add $\frac{x-2}{3}$ and $\frac{4x}{7}$ together. *Ans.* $\frac{19x-14}{21}$.

5. Add $x + \frac{x-2}{3}$ to $3x + \frac{2x-3}{4}$.

Ans. $4x + \frac{10x-17}{12}$.

6. It is required to add $4x$, $\frac{5x^2}{2a}$, and $\frac{x+a}{2x}$ together.

Ans. $4x + \frac{5x^3 + ax + a^2}{2ax}$.

7. It is required to add $\frac{2x}{3}$, $\frac{7x}{4}$, and $\frac{2x+1}{5}$ together.

Ans. $2x + \frac{49x+12}{60}$.

8. It is required to add $4x$, $\frac{7x}{9}$, and $2 + \frac{x}{5}$ together.

Ans. $4x + \frac{44x+90}{45}$.

9. It is required to add $3x + \frac{2x}{5}$ and $x - \frac{8x}{9}$ together.

Ans. $3x + \frac{23x}{45}$.

10. What is the sum of $\frac{a-x}{a-b}$, $\frac{c}{a+b}$, and $\frac{d}{a+x}$.

$$Ans.\ \frac{a^3 - ax^2 + a^2b - bx^2 + a^2c + acx - abc - bcx + a^2d - b^2d}{a^3 - b^2a + a^2x - b^2x}$$

$$= \frac{a^3 + a^2(b+c+d) - a(x^2 - cx + bc) - b(x^2 + cx + bd)}{a^3 + a^2x - ab^2 - b^2x}.$$

CASE VI.

75. To subtract one fractional quantity from another.

RULE.

I. *Reduce the fractions to a common denominator.*

II. *Subtract the numerator of the subtrahend from the numerator of the minuend, and place the difference over the common denominator.*

EXAMPLES.

1. Find the difference of the fractions $\frac{x-a}{2b}$ and $\frac{2a-4x}{3c}$

$$\left.\begin{aligned}\text{Here,}\quad (x-a)\times 3c &= 3cx-3ac\\ (2a-4x)\times 2b &= 4ab-8bx\end{aligned}\right\}\text{ the numerators.}$$

$$\text{And,}\qquad 2b\times 3c = 6bc\qquad \text{the common denominator.}$$

$$\text{Hence,}\quad \frac{3cx-3ac}{6bc}-\frac{4ab-8bx}{6bc}=\frac{3cx-3ac-4ab+8bx}{6bc}.$$

2. Required the difference of $\frac{12x}{7}$ and $\frac{3x}{5}$. *Ans.* $\frac{39x}{35}$.

3. Required the difference of $5y$ and $\frac{3y}{8}$. *Ans.* $\frac{37y}{8}$.

4. Required the difference of $\frac{3x}{7}$ and $\frac{2x}{9}$. *Ans.* $\frac{13x}{63}$.

5. Required the difference of $\frac{x+a}{b}$ and $\frac{c}{d}$.

Ans. $\frac{dx+ad-bc}{bd}$.

6. Required the difference of $\frac{3x+a}{5b}$ and $\frac{2x+7}{8}$.

Ans. $\frac{24x+8a-10bx-35b}{40b}$.

7. Required the difference of $3x+\frac{x}{b}$ and $x-\frac{x-a}{c}$.

Ans. $2x+\frac{cx+bx-ab}{bc}$.

CASE VII.

76. To multiply fractional quantities together.

RULE.

If the quantities to be multiplied are mixed, reduce them to a fractional form; then multiply the numerators together for a numerator and the denominators together for a denominator.

EXAMPLES.

1. Multiply $a + \frac{bx}{a}$ by $\frac{c}{d}$.

First, - - - $a + \frac{bx}{a} = \frac{a^2 + bx}{a}$;

Hence, - $\frac{a^2 + bx}{a} \times \frac{c}{d} = \frac{a^2c + bcx}{ad}$;

2. Required the product of $\frac{3x}{2}$ and $\frac{3a}{b}$. *Ans.* $\frac{9ax}{2b}$.

3. Required the product of $\frac{2x}{5}$ and $\frac{3x^2}{2a}$. *Ans.* $\frac{3x^3}{5a}$.

4. Find the continued product of $\frac{2x}{a}$, $\frac{3ab}{c}$, and $\frac{3ac}{2b}$.

Ans. $9ax$.

5. It is required to find the product of $b + \frac{bx}{a}$ and $\frac{a}{x}$.

Ans. $\frac{ab + bx}{x}$.

6. Required the product of $\frac{x^2 - b^2}{bc}$ and $\frac{x^2 + b^2}{b + c}$.

Ans. $\frac{x^4 - b^4}{b^2c + bc^2}$.

7. Required the product of $x + \frac{x + 1}{a}$, and $\frac{x - 1}{a + b}$.

Ans. $\frac{ax^2 - ax + x^2 - 1}{a^2 + ab}$.

8. Required the product of $a + \frac{ax}{a - x}$ by $\frac{a^2 - x^2}{x + x^2}$.

Ans. $\frac{a^2(a + x)}{x\,(1 + x)}$.

CASE VIII.

77. To divide one fractional quantity by another.

RULE.

Reduce the mixed quantities, if there are any, to a fractional form; then invert the terms of the divisor and multiply the fractions together as in the last case.

EXAMPLES.

1. Divide - - - $a - \dfrac{b}{2c}$ by $\dfrac{f}{g}$.

$$a - \frac{b}{2c} = \frac{2ac - b}{2c}$$

Hence, $\quad a - \dfrac{b}{2c} \div \dfrac{f}{g} = \dfrac{2ac-b}{2c} \times \dfrac{g}{f} = \dfrac{2acg - bg}{2cf}$.

2. Let $\dfrac{7x}{5}$ be divided by $\dfrac{12}{13}$. *Ans.* $\dfrac{91x}{60}$.

3. Let $\dfrac{4x^2}{7}$ be divided by $5x$. *Ans.* $\dfrac{4x}{35}$.

4. Let $\dfrac{x+1}{6}$ be divided by $\dfrac{2x}{3}$. *Ans.* $\dfrac{x+1}{4x}$.

5. Let $\dfrac{x}{x-1}$ be divided by $\dfrac{x}{2}$. *Ans.* $\dfrac{2}{x-1}$.

6. Let $\dfrac{5x}{3}$ be divided by $\dfrac{2a}{3b}$. *Ans.* $\dfrac{5bx}{2a}$.

7. Let $\dfrac{x-b}{8cd}$ be divided by $\dfrac{3cx}{4d}$. *Ans.* $\dfrac{x-b}{6c^2x}$.

8. Let $\dfrac{x^4 - b^4}{x^2 - 2bx + b^2}$ be divided by $\dfrac{x^2+bx}{x-b}$.

Ans. $x + \dfrac{b^2}{x}$.

9. Divide $\dfrac{ax-1}{1-x}$ by $\dfrac{a}{(1-x^2)}$ $(1-x)(1+x)$ *Ans.* $\dfrac{ax(1+x)-x-1}{a}$.

10. Divide $\dfrac{a+1}{a-1}$ by $\dfrac{1+a}{(1-a^2)}$ $=(1-a)(1+a)$ *Ans.* $-(1+a)$.

If we have a fraction of the form

$$\frac{a}{b} = c,$$

we may observe that

$$\frac{-a}{b} = -c, \text{ also } \frac{a}{-b} = -c \text{ and } \frac{-a}{-b} = c; \text{ that is,}$$

The sign of the quotient will be changed by changing the sign either of the numerator or denominator, but will not be affected by changing the signs of both the terms.

78. We will add but two propositions more on the subject of fractions.

If the same number be added to each of the terms of a proper fraction, the new fraction resulting from this addition will be greater than the first; but if it be added to the terms of an improper fraction, the resulting fraction will be less than the first.

Let the fraction be expressed by $\frac{a}{b}$, and suppose $a < b$.

Let m represent the number to be added to each term: then the new fraction becomes $\frac{a+m}{b+m}$.

In order to compare the two fractions, they must be reduced to the same denominator, which gives for

the first fraction, $\frac{a}{b} = \frac{ab + am}{b^2 + bm}$

and for the new fraction, $\frac{a+m}{b+m} = \frac{ab + bm}{b^2 + bm}$.

Now, the denominators being the same, that fraction will be the greatest which has the greater numerator. But the two numerators have a common part ab, and the part bm of the second is greater than the part am of the first, since $b > a$: hence

$$ab + bm > ab + am;$$

that is, the second fraction is greater than the first.

If the given fraction is improper, that is, if $a > b$, it is plain that the numerator of the second fraction will be less than that of the first, since bm would then be less than am.

If the same number be subtracted from each term of a proper fraction, the value of the fraction will be diminished; but if it be subtracted from the terms of an improper fraction, the value of the fraction will be increased.

Let the fraction be expressed by $\frac{a}{b}$, and denote the number to be subtracted by m.

Then, $\frac{a-m}{b-m} =$ the new fraction.

By reducing to the same denominator, we have,

$$\frac{a}{b} = \frac{ab - am}{b^2 - bm};$$

and

$$\frac{a - m}{b - m} = \frac{ab - bm}{b^2 - bm}.$$

Now, if we suppose $a < b$, then $am < bm$; and if $am < bm$, then will

$$ab - am > ab - bm:$$

that is, the new fraction will be less than the first.

If $a > b$, that is, if the fraction is improper, then

$$am > bm, \text{ and } ab - am < ab - bm,$$

that is, the new fraction will be greater than the first.

GENERAL EXAMPLES.

1. Add $\frac{1 + x^2}{1 - x^2}$ to $\frac{1 - x^2}{1 + x^2}$. *Ans.* $\frac{2(1 + x^4)}{1 - x^4}$.

2. Add $\frac{1}{1 + x}$ to $\frac{1}{1 - x}$. *Ans.* $\frac{2}{1 - x^2}$.

3. From $\frac{a + b}{a - b}$ take $\frac{a - b}{a + b}$. *Ans.* $\frac{4ab}{a^2 - b^2}$.

4. From $\frac{1 + x^2}{1 - x^2}$ take $\frac{1 - x^2}{1 + x^2}$. *Ans.* $\frac{4x^2}{1 - x^4}$.

5. Multiply $\frac{x^2 - 9x + 20}{x^2 - 6x}$ by $\frac{x^2 - 13x + 42}{x^2 - 5x}$.

   Ans. $\frac{x^2 - 11x + 28}{x^2}$.

6. Multiply $\frac{x^4 - b^4}{x^2 + 2bx + b^2}$ by $\frac{x^2 + bx}{x - b}$. *Ans.* $x^3 + b^2x$.

7. Divide $\frac{a + x}{a - x} + \frac{a - x}{a + x}$ by $\frac{a + x}{a - x} - \frac{a - x}{a + x}$.

   Ans. $\frac{a^2 + x^2}{2ax}$.

8. Divide $1 + \frac{n - 1}{n + 1}$ by $1 - \frac{n - 1}{n + 1}$. *Ans.* n.

CHAPTER IV.

OF EQUATIONS OF THE FIRST DEGREE.

79. AN *Equation* is the algebraic expression of two equal quantities with the sign of equality placed between them. Thus,

$$x = a + b$$

is an equation, in which x is equal to the sum of a and b.

80. By the definition, every equation is composed of two parts, separated from each other by the sign $=$. The part on the left of the sign, is called the *first member*, and the part on the right, is called the *second member;* and each member may be composed of one or more terms.

81. Every equation may be regarded as the enunciation, in algebraic language, of a particular problem. Thus, the equation

$$x + x = 30,$$

is the algebraic enunciation of the following problem:

To find a number which, being added to itself, shall give a sum equal to 30.

Were it required to solve this problem, we should first express it in algebraic language, which would give the equation

$$x + x = 30,$$

by adding x to itself, - - $2x = 30,$

and dividing by 2, - - $x = 15.$

Hence we see that the solution of a problem by algebra, consists of two distinct parts: viz., the *statement*, and the *solution* of an equation.

The STATEMENT consists *in finding an equation which shall express the relation between the known and unknown quantities of the problem.*

The SOLUTION of the equation consists *in finding such a value for the unknown quantity as being substituted for it in the equation will satisfy it; that is, make the first member equal to the second.*

82. An equation is said to be *verified*, when such a value is substituted for the unknown quantity as will prove the two members of the equation to be equal to each other.

83. Equations are divided into classes, with reference to the highest exponent with which the unknown quantity is affected.

An equation which contains only the first power of the unknown quantity, is called an equation of the *first degree:* and generally, the *degree* of an equation is determined by the greatest of the exponents with which the unknown quantity is affected, without reference to other terms which may contain the unknown quantity raised to a less power. Thus,

$ax + b = cx + d$ is an equation of the 1st degree.

$2x^2 - 3x = 5 - 2x^2$ is an equation of the 2d degree.

$4x^3 - 5x^2 = 3x + 11$ is an equation of the 3d degree.

If more than one unknown quantity enters into an equation, its degree is determined by the greatest sum of the exponents with which the unknown quantities are affected in any of its terms. Thus,

$xy + bcx = d^4$ is of the second degree.

$xyz^2 + cx^2 = a^5$ is of the fourth degree.

84. Equations are also distinguished as *numerical equations* and *literal equations.* The first are those which contain numbers only, with the exception of the unknown quantity, which is always denoted by a letter. Thus,

$$4x - 3 = 2x + 5, \quad 3x^2 - x = 8,$$

are numerical equations. They are the algebraical translation of problems, in which the known quantities are particular numbers.

A literal equation is one in which a part, or all of the known quantities, are represented by letters. Thus,

$$bx^2 + ax - 3x = 5, \text{ and } cx + dx^2 = c + f,$$

are literal equations.

85. It frequently occurs in Algebra, that the algebraic sign $+$ or $-$, which is written, is not the true sign of the term before which it is placed. Thus, if it were required to subtract $-b$ from a, we should write

$$a - (-b) = a + b.$$

Here the true sign of the second term of the binomial is plus, although its algebraic sign, which is written in the first member of the equation, is $-$. This minus sign, operating upon the sign of b, which is also negative, produces a plus sign for b in the result. The sign which results, after combining the algebraic sign with the sign of the quantity, *is called the essential sign of the term*, and is often different from the algebraic sign.

By considering the nature of an equation, we perceive that it must possess the three following properties:

1st. The two members are composed of quantities of the same kind.

2d. The two members are equal to each other.

3d. The essential sign of the two members must be the same.

86. An axiom is a self-evident proposition. We may here state the following:

1. If equal quantities be added to both members of an equation, the equality of the members will not be destroyed.

2 If equal quantities be subtracted from both members of an equation, the equality will not be destroyed.

3. If both members of an equation be multiplied by the same number, the equality will not be destroyed.

4. If both members of an equation be divided by the same number, the equality will not be destroyed.

Solution of Equations of the First Degree.

87. The *transformation* of an equation is any operation by which we change the form of the equation without affecting the equality of its members

First Transformation.

88. When some of the terms of an equation are fractional, to reduce the equation to one in which the terms shall be entire.

Take the equation,

$$\frac{2x}{3} - \frac{3}{4}x + \frac{x}{6} = 11.$$

First, reduce all the fractions to the same denominator, by the known rule; the equation then becomes

$$\frac{48x}{72} - \frac{54x}{72} + \frac{12x}{72} = 11.$$

If now, both members of this equation be multiplied by 72, the equality of the members will be preserved, and the common denominator will disappear; and we shall have

$$48x - 54x + 12x = 792;$$

or dividing by 6, $\quad 8x - 9x + 2x = 132.$

89. The last equation could have been found in another manner by employing the least common multiple of the denominators.

The *common multiple* of two or more numbers is any number which each will divide without a remainder; and the *least* common multiple, is the least number which can be so divided.

The least common multiple can generally be found by inspection. Thus, 24 is the least common multiple of 4, 6, and 8; and 12 is the least common multiple of 3, 4, and 6.

Take the last equation,

$$\frac{2x}{3} - \frac{3}{4}x + \frac{x}{6} = 11.$$

We see that 12 is the least common multiple of the denominators, and if we multiply each term of the equation by 12, dividing at the same time by the denominators, we obtain

$$8x - 9x + 2x = 132,$$

the same equation as before found.

90. Hence, to transform an equation involving fractional terms to one involving only entire terms, we have the following

RULE.

Form the least common multiple of all the denominators, and then multiply every term of the equation by it, reducing at the same time the fractional to entire terms.

EXAMPLES.

1. Reduce $\frac{x}{5} + \frac{x}{4} - 3 = 20$, to an equation involving entire terms.

We see, at once, that the least common multiple is 20, by which each term of the equation is to be multiplied.

Now, $$\frac{x}{5} \times 20 = x \times \frac{20}{5} = 4x,$$

and $$\frac{x}{4} \times 20 = x \times \frac{20}{4} = 5x;$$

that is, we reduce the fractional to entire terms, *by multiplying the numerator by the quotient of the common multiple divided by the denominator, and omitting the denominators.*

Hence, the transformed equation is

$$4x + 5x - 60 = 400.$$

2. Reduce $\frac{x}{5} + \frac{x}{7} - 4 = 3$ to an equation involving only entire terms. *Ans.* $7x + 5x - 140 = 105$.

3. Reduce $\frac{a}{b} - \frac{c}{d} + f = g$ to an equation involving only entire terms. *Ans.* $ad - bc + bdf = bdg$.

4. Reduce the equation

$$\frac{ax}{b} - \frac{2c^2x}{ab} + 4a = \frac{4bc^2x}{a^3} - \frac{5a^3}{b^2} + \frac{2c^2}{a} - 3b$$

to one involving only entire terms.

Ans. $a^4bx - 2a^2bc^2x + 4a^4b^2 = 4b^3c^2x - 5a^6 + 2a^2b^2c^2 - 3a^3b^3$.

Second Transformation.

91. When the two members of an equation are entire polynomials to transpose certain terms from one member to the other.

Take for example the equation $5x - 6 = 8 + 2x$.

If, in the first place we subtract $2x$ from both members, the equality will not be destroyed, and we have $\Big\}$ $5x - 6 - 2x = 8 + 2x - 2x;$

or, by reducing the terms in the second member, $\Big\}$ $5x - 6 - 2x = 8.$

Whence we see that the term $2x$, which was additive in the second member, becomes subtractive in the first.

In the second place, if we add 6 to both members, the equality will still exist, and we have $\left.\right\}$ $5x - 6 - 2x + 6 = 8 + 6$;

or, since -6 and $+6$ destroy each other $5x - 2x = 8 + 6$.

Hence, the term which was subtractive in the first member, passes into the second member with the sign plus.

For a second example, take the equation

$$ax + b = d - cx.$$

If we add cx to both members and subtract b, the equation becomes $\left.\right\}$ $ax + b + cx - b = d - cx + cx - b$;

or reducing - - - $ax + cx = d - b$.

Hence, we have the following principle:

Any term of an equation may be transposed from one member to the other by changing its sign.

92. We will now apply the preceding principles to the resolution of equations.

1. Take the equation $4x - 3 = 2x + 5$.

By transposing the terms -3 and $2x$, it becomes

$$4x - 2x = 5 + 3;$$

and by reducing $2x = 8$:

dividing by 2 $x = \frac{8}{2} = 4$.

Now, if 4 be substituted in the place of x in the given equation, it becomes

$$4 \times 4 - 3 = 2 \times 4 + 5,$$

that is, $13 = 13$.

Hence, 4 is the true value of x; for, being substituted for x in the given equation, that equation is verified.

2. For a second example, take the equation

$$\frac{5x}{12} - \frac{4x}{3} - 13 = \frac{7}{8} - \frac{13x}{6}.$$

By making the denominators disappear, we have

$$10x - 32x - 312 = 21 - 52x$$

by transposing $\quad 10x - 32x + 52x = 21 + 312$

by reducing $\quad 30x = 333$

dividing by 30 $\quad x = \frac{333}{30} = \frac{111}{10} = 11.1;$

a result which, being substituted for x, will verify the given equation.

3. For a third example let us take the equation

$$(3a - x)(a - b) + 2ax = 4b(x + a).$$

It is first necessary to perform the multiplications indicated, in order to reduce the two members to polynomials, and thus be able to disengage the unknown quantity x from the known quantities. Having performed the multiplications, the equation becomes,

$$3a^2 - ax - 3ab + bx + 2ax = 4bx + 4ab;$$

by transposing $\quad -ax + bx + 2ax - 4bx = 4ab + 3ab - 3a^2,$

by reducing $\quad ax - 3bx = 7ab - 3a^2;$

or, (Art. 48), $\quad (a - 3b)x = 7ab - 3a^2.$

Dividing both members by $a - 3b$, we find

$$x = \frac{7ab - 3a^2}{a - 3b}.$$

93. Hence, in order to resolve any equation of the first degree, we have the following general

RULE.

I. *If the equation contains fractional terms, reduce it to one in which all the terms shall be entire, and then transpose all the terms affected with the unknown quantity into the first member, and all the known terms into the second.*

II. *Reduce to a single term all the terms involving the unknown quantity: this term will be composed of two factors, one of which will be the unknown quantity, and the other all its co-efficients connected by their respective signs.*

III. *Then divide both members of the equation by the multiplier of the unknown quantity.*

EXAMPLES.

1. Given $3x - 2 + 24 = 31$ to find x. *Ans.* $x = 3$.

2. Given $x + 18 = 3x - 5$ to find x. *Ans.* $x = 11\frac{1}{2}$.

3. Given $6 - 2x + 10 = 20 - 3x - 2$ to find x. *Ans.* $x = 2$.

4. Given $x + \frac{1}{2}x + \frac{1}{3}x = 11$ to find x. *Ans.* $x = 6$.

5. Given $2x - \frac{1}{2}x + 1 = 5x - 2$ to find x. *Ans.* $x = \frac{6}{7}$.

6. Given $3ax + \frac{a}{2} - 3 = bx - a$ to find x.

Ans. $x = \dfrac{6 - 3a}{6a - 2b}$.

7. Given $\dfrac{x-3}{2} + \dfrac{x}{3} = 20 - \dfrac{x-19}{2}$ to find x.

Ans. $x = 23\frac{1}{4}$.

8. Given $\dfrac{x+3}{2} + \dfrac{x}{3} = 4 - \dfrac{x-5}{4}$ to find x.

Ans. $x = 3\frac{6}{13}$.

9. Given $\dfrac{ax-b}{4} + \dfrac{a}{3} = \dfrac{bx}{2} - \dfrac{bx-a}{3}$ to find x.

Ans. $x = \dfrac{3b}{3a - 2b}$.

10. Given $\dfrac{3ax}{c} - \dfrac{2bx}{d} - 4 = f$, to find x.

Ans. $x = \dfrac{cdf + 4cd}{3ad - 2bc}$.

11. Given $\dfrac{8ax-b}{7} - \dfrac{3b-c}{2} = 4 - b$, to find x.

Ans. $x = \dfrac{56 + 9b - 7c}{16a}$.

12. Given $\dfrac{x}{5} - \dfrac{x-2}{3} + \dfrac{x}{2} = \dfrac{13}{3}$, to find x.

Ans. $x = 10$.

13. Given $\dfrac{x}{a} - \dfrac{x}{b} + \dfrac{x}{c} - \dfrac{x}{d} = f$, to find x.

Ans. $x = \dfrac{abcdf}{bcd - acd + adb - abc}$.

14. Given $x - \frac{3x-5}{13} + \frac{4x-2}{11} = x + 1$, to find x.

Ans. $x = 6$.

15. Given $\frac{x}{7} - \frac{8x}{9} - \frac{x-3}{5} = -12\frac{29}{45}$, to find x.

Ans. $x = 14$.

16. Given $2x - \frac{4x-2}{5} = \frac{3x-1}{2}$, to find x.

Ans. $x = 3$.

17. Given $3x + \frac{bx-d}{3} = x + a$, to find x.

Ans. $x = \frac{3a+d}{6+b}$.

18. Find the value of x in the equation

$$\frac{(a+b)(x-b)}{a-b} - 3a = \frac{4ab-b^2}{a+b} - 2x + \frac{a^2-bx}{b}.$$

Ans. $x = \frac{a^4 + 3a^3b + 4a^2b^2 - 6ab^3 + 2b^4}{2b(2a^2+ab-b^2)}$.

Questions producing Equations of the First Degree, involving but one Unknown Quantity.

94. It has already been observed (Art. 81), that the solution of a problem by Algebra, consists of two distinct parts.

1st. The statement; and

2d. The solution of the equation.

We have already explained the methods of solving the equation; and it only remains to point out the best manner of making the statement.

This part cannot, like the second, be subjected to any well-defined rule. Sometimes the enunciation of the problem furnishes the equation immediately; and sometimes it is necessary to discover, from the enunciation, new conditions from which an equation may be formed.

The conditions enunciated are called *explicit conditions*, and those which are deduced from them, *implicit conditions*.

In almost all cases, however, we are enabled to discover the equation by applying the following

RULE.

Represent the unknown quantity by one of the final letters of the alphabet, and then indicate, by means of the algebraic signs, the same operations on the known and unknown quantities, as would verify the value of the unknown quantity, were such value known.

QUESTIONS.

1. Find a number such, that the sum of one half, one third, and one fourth of it, augmented by 45, shall be equal to 448.

Let the required number be denoted by - - - - x.

Then, one half of it will be denoted by - - - - $\frac{x}{2}$.

one third of it - - - - by - - - - $\frac{x}{3}$.

one fourth of it - - - - by - - - - $\frac{x}{4}$.

And by the conditions, $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+45=448$.

Now, by subtracting 45 from both members,

$$\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=403.$$

By making the terms of the equation entire, we obtain

$$6x+4x+3x=4836;$$

or - - $13x=4836$.

Hence - $x=\frac{4836}{13}=372$.

Let us see if this value will verify the equation of the problem. We have

$$\frac{372}{2}+\frac{372}{3}+\frac{372}{4}+45=186+124+93+45=448.$$

2. What number is that whose third part exceeds its fourth, by 16.

Let the required number be represented by x. Then

$\frac{1}{3}x=$ the third part.

$\frac{1}{4}x=$ the fourth part.

And by the question $\frac{1}{3}x - \frac{1}{4}x = 16.$

or, - - - - - $4x - 3x = 192.$

$$x = 192.$$

Verification.

$$\frac{192}{3} - \frac{192}{4} = 64 - 48 = 16.$$

3. Out of a cask of wine which had leaked away a third part, 21 gallons were afterward drawn, and the cask being then gauged, appeared to be half full: how much did it hold?

Suppose the cask to have held x gallons.

Then, $\frac{x}{3} =$ what leaked away.

And $\frac{x}{3} + 21 =$ what leaked out, and what was drawn.

Hence, $\frac{x}{3} + 21 = \frac{1}{2}x$ by the question.

or $2x + 126 = 3x.$

or $-x = -126.$

or $x = 126,$

by changing the signs of both members, which does not destroy their equality.

Verification.

$$\frac{126}{3} + 21 = 42 + 21 = \frac{126}{2} = 63.$$

4. A fish was caught whose tail weighed 9*lb.*; his head weighed as much as his tail and half his body, and his body weighed as much as his head and tail together: what was the weight of the fish?

Let - - $2x =$ the weight of the body.

Then - - $9 + x =$ weight of the head.

And since the body weighed as much as both head and tail.

$$2x = 9 + 9 + x$$

or - - - $2x - x = 18$

and - - - $x = 18.$

Verification.

$$
\begin{array}{rrl}
 & 2x = 36\,lb = & \text{weight of the body.} \\
 & 9 + x = 27\,lb = & \text{weight of the head.} \\
 & 9\,lb = & \text{weight of the tail.} \\
\text{Hence,} & \overline{72\,lb} = & \text{weight of the fish.}
\end{array}
$$

5. A person engaged a workman for 48 days. For each day that he labored he received 24 cents, and for each day that he was idle, he paid 12 cents for his board. At the end of the 48 days, the account was settled, when the laborer received 504 cents. *Required the number of working days, and the number of days he was idle.*

If these two numbers were known, by multiplying them respectively by 24 and 12, then subtracting the last product from the first, the result would be 504. Let us indicate these operations by means of algebraic signs.

$$
\begin{array}{lrl}
\text{Let} & x = & \text{the number of working days.} \\
\text{Then} & 48 - x = & \text{the number of idle days.} \\
 & 24 \times x = & \text{the amount earned, and} \\
 & 12(48 - x) = & \text{the amount paid for his board.}
\end{array}
$$

$$
\begin{array}{ll}
\text{Then} & 24x - 12(48 - x) = 504 \quad \text{what he received.} \\
\text{or} & 24x - 576 + 12x = 504. \\
\text{or} & 36x = 504 + 576 = 1080 \\
\text{and} & x = \dfrac{1080}{36} = 30 \quad \text{the working days.} \\
\text{whence,} & 48 - 30 = 18 \quad \text{the idle days.}
\end{array}
$$

Verification.

Thirty day's labor, at 24 cents a day, amounts to - - - - - - - - $30 \times 24 = 720$ cts.

And 18 days' board, at 12 cents a day, amounts to - - - - - - - - $18 \times 12 = 216$ cts.

And the amount received is their difference $\overline{504}$.

General Solution.

Let, $n =$ the whole number of working and idle days,
$a =$ the amount received for each day he worked,
$b =$ the amount paid for his board, for each idle day,
$c =$ the balance due, or the result of the account,
$x =$ the number of working days,
$n - x =$ the number of idle days.
Then, $ax =$ what he earned;
and, $b(n - x) =$ the amount deducted for board.

The equation of the problem will then be,

$$ax - b(n - x) = c$$

whence

$$ax - bn + bx = c$$

$$(a + b)x = c + bn$$

$$x = \frac{c + bn}{a + b}$$

and consequently,

$$n - x = n - \frac{c + bn}{a + b} = \frac{an + bn - c - bn}{a + b}$$

or

$$n - x = \frac{an - c}{a + b}.$$

6. A fox, pursued by a greyhound, has a start of 60 leaps. He makes 9 leaps while the greyhound makes but 6; but 3 leaps of the greyhound are equivalent to 7 of the fox. How many leaps must the greyhound make to overtake the fox?

From the enunciation, it is evident that, the distance to be passed over by the greyhound, is equal to the 60 leaps of the fox, plus the distance which the fox runs after the greyhound starts in pursuit.

Let $x =$ the number of leaps made by the greyhound from the time of starting till he overtakes the fox.

Now, since the fox makes 9 leaps while the greyhound makes 6, the fox will make $1\frac{1}{2}$, or $\frac{3}{2}$ leaps while the greyhound makes 1; and, therefore, while the greyhound makes x leaps, the fox will make $\frac{3}{2}x$ leaps. Hence,

$$60 + \frac{3}{2}x =$$

the number of leaps made by the fox, in passing over the entire distance.

It might, at first, be supposed that the equation of the problem would be obtained by placing this number equal to x; but in doing so, a manifest error would be committed; for the leaps of the greyhound are greater than those of the fox, and we should thus equate numbers referred to different units. Hence, it is necessary to express the leaps of the fox by means of those of the greyhound, or reciprocally.

Now, according to the enunciation, 3 leaps of the greyhound are equivalent to 7 leaps of the fox; and hence, 1 leap of the greyhound is equivalent to $\frac{7}{3}$ leaps of the fox; consequently, x leaps of the greyhound are equivalent to $\frac{7x}{3}$ of the fox: that is, had the leaps of the greyhound been no longer than those of the fox, he would have made $\frac{7x}{3}$ leaps instead of x leaps.

Hence the true equation is, $\frac{7x}{3} = 60 + \frac{3}{2}x$;

or, by making the terms entire $14x = 360 + 9x$,

whence - - - - - - $5x = 360$ and $x = 72$.

Therefore, the greyhound will make 72 leaps to overtake the fox, and during this time the fox will make $72 \times \frac{3}{2} = 108$.

Verification.

The 72 leaps of the greyhound are equivalent to

$\frac{72 \times 7}{3} = 168$ leaps of the fox = the whole distance.

And $60 + 108 = 168$, the leaps which the fox made from the beginning.

7. A can do a piece of work alone in 10 days, and B in 13 days: in what time can they do it if they work together?

Denote the time by x, and the work to be done by 1. Then in 1 day A could do $\frac{1}{10}$ of the work, and B could do $\frac{1}{13}$ of

it; and in x days A could do $\frac{x}{10}$ of the work, and B $\frac{x}{13}$: hence, by the conditions of the question,

$$\frac{x}{10}+\frac{x}{13}=1,$$

which gives $$13x+10x=130:$$

hence, $$23x=130, \quad x=\frac{130}{23}=5\tfrac{15}{23} \text{ days.}$$

8. Divide \$1000 between A, B, and C, so that A shall have \$72 more than B, and C \$100 more than A.

Ans. A's share = \$324, B's = \$252, C's = \$424.

9. A and B play together at cards. A sits down with \$84 and B with \$48. Each loses and wins in turn, when it appears that A has five times as much as B. How much did A win? *Ans.* \$26.

10. A person dying leaves half of his property to his wife, one sixth to each of two daughters, one twelfth to a servant, and the remaining \$600 to the poor: what was the amount of his property? *Ans.* \$7200.

11. A father leaves his property, amounting to \$2520, to four sons, A, B, C, and D. C is to have \$360, B as much as C and D together, and A twice as much as B less \$1000: how much does A, B, and D, receive?

Ans. A \$760, B \$880, D \$520.

12. An estate of \$7500 is to be divided between a widow, two sons, and three daughters, so that each son shall receive twice as much as each daughter, and the widow herself \$500 more than all the children: what was her share, and what the share of each child?

Ans. Widow's share \$4000. Each son \$1000. Each daughter \$500.

13. A company of 180 persons consists of men, women, and children. The men are 8 more in number than the women, and the children 20 more than the men and women together: how many of each sort in the company?

Ans. 44 men, 36 women, 100 children.

14. A father divides $2000 among five sons, so that each elder should receive $40 more than his next younger brother: what is the share of the youngest? *Ans.* $320.

15. A purse of $2850 is to be divided among three persons, A, B, and C; A's share is to be to B's as 6 to 11, and C is to have $300 more than A and B together: what is each one's share? *Ans.* A's $450, B's $825, C's $1575.

16. Two pedestrians start from the same point; the first steps twice as far as the second, but the second makes 5 steps while the first makes but one. At the end of a certain time they are 300 feet apart. Now, allowing each of the longer paces to be 3 feet, how far will each have travelled?

Ans. 1st, 200 *feet;* 2d, 500.

17. Two carpenters, 24 journeymen, and 8 apprentices, received at the end of a certain time $144. The carpenters received $1 per day, each journeyman half a dollar, and each apprentice 25 cents: how many days were they employed?

Ans. 9 *days.*

18. A capitalist receives a yearly income of $2940: four fifths of his money bears an interest of 4 per cent., and the remainder of 5 per cent.: how much has he at interest? *Ans.* $70000.

19. A cistern containing 60 gallons of water has three unequal cocks for discharging it; the largest will empty it in one hour, the second in two hours, and the third in three: in what time will the cistern be emptied if they all run together?

Ans. $32\frac{8}{11}$ *min.*

20. In a certain orchard $\frac{1}{2}$ are apple-trees, $\frac{1}{4}$ peach-trees, $\frac{1}{6}$ plum-trees, 120 cherry-trees, and 80 pear-trees: how many trees in the orchard? *Ans.* 2400.

21. A farmer being asked how many sheep he had, answered that he had them in five fields; in the 1st he had $\frac{1}{4}$, in the 2d $\frac{1}{6}$, in the 3d $\frac{1}{8}$, in the 4th $\frac{1}{12}$, and in the 5th 450: how many had he? *Ans.* 1200.

22. My horse and saddle together are worth $132, and the horse is worth ten times as much as the saddle: what is the value of the horse? *Ans.* $120.

23. The rent of an estate is this year 8 per cent. greater than it was last. This year it is \$1890: what was it last year?
Ans. \$1750.

24. What number is that from which, if 5 be subtracted, $\frac{2}{3}$ of the remainder will be 40? *Ans.* 65.

25. A post is $\frac{1}{4}$ in the mud, $\frac{1}{3}$ in the water, and ten feet above the water: what is the whole length of the post?
Ans. 24 *feet.*

26. After paying $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I had 66 guineas left in my purse: how many guineas were in it at first?
Ans. 120.

27. A person was desirous of giving 3 pence apiece to some beggars, but found he had not money enough in his pocket by 8 pence; he therefore gave them each two pence and had 3 pence remaining: required the number of beggars. *Ans.* 11.

28. A person in play lost $\frac{1}{4}$ of his money, and then won 3 shillings; after which he lost $\frac{1}{3}$ of what he then had; and this done, found that he had but 12 shillings remaining: what had he at first? *Ans.* 20*s*.

29. Two persons, A and B, lay out equal sums of money in trade; A gains \$126, and B loses \$87, and A's money is now double of B's: what did each lay out? *Ans.* \$300.

30. A farmer bought a basket of eggs, and offered them at 7 cents a dozen. But before he sold any, 5 dozen were broken by a careless boy, for which he was paid. He then sold the remainder at 8 cents a dozen, and received as much as he would have got for the whole at the first price. How many eggs had he in his basket? *Ans.* 40 dozen.

31. A person goes to a tavern with a certain sum of money in his pocket, where he spends 2 shillings; he then borrows as much money as he had left, and going to another tavern, he there spends 2 shillings also; then borrowing again as much money as was left, he went to a third tavern, where likewise he spent 2 shillings and borrowed as much as he had left; and again spending 2 shillings at a fourth tavern, he then had nothing remaining. What had he at first?
Ans. 3*s*. 9*d*

Of Equations of the First Degree, involving two or more Unknown Quantities.

95. Although several of the previous questions contained in their enunciation more than one unknown quantity, we have nevertheless resolved them all by employing but one symbol. The reason of this is, that we have been able, from the conditions of the enunciation, to represent the other unknown quantities by means of this symbol and known quantities; but this cannot be done in all problems containing more than one unknown quantity.

To explain the methods of resolving problems of this kind, let us take some of those which have been resolved by means of one unknown quantity.

1. Given the sum of two numbers equal to a, and their difference equal to b; it is required to find the numbers.

Let $x =$ the greater, and y the less number.

Then by the conditions $\quad x + y = a$;

and $\quad x - y = b$.

By adding (Art. 86, Ax. 1), $\quad 2x = a + b$.

By subtracting (Art. 86, Ax. 2), $\quad 2y = a - b$.

Each of these equations contains but one unknown quantity.

From the first we obtain $\quad x = \dfrac{a+b}{2}$.

And from the second $\quad y = \dfrac{a-b}{2}$.

Verification.

$$\frac{a+b}{2} + \frac{a-b}{2} = \frac{2a}{2} = a; \text{ and } \frac{a+b}{2} - \frac{a-b}{2} = \frac{2b}{2} = b.$$

2. A person engaged a workman a number of days, denoted by n. For each day that he labored he was to receive a cents, and for each day that he was idle he was to pay b cents for his board. At the end of the n days, the account was settled, when the laborer received c cents. Required the number of working days and the number of days he was idle.

Let $\quad x =$ the number of working days.

$\quad y =$ the number of idle days.

Then, $\quad ax =$ what he earned,

and $\quad by =$ what he paid for his board;

and by the question, we have $\begin{cases} x + y = n \\ ax - by = c. \end{cases}$

It has already been shown that the two members of an equation can be multiplied by the same number, without destroying the equality; therefore, multiply both members of the first equation by b, the co-efficient of y in the second, and we have

the equation - - - - - - $bx + by = bn,$

which, added to the second - $ax - by = c,$

gives - - - - - - - $ax + bx = bn + c.$

Whence - - - - - - - $x = \dfrac{bn + c}{a + b}$

In like manner, multiplying the two members of the first equation by a, the co-efficient of x in the second, it becomes

$ax + ay = an;$

from which, subtract the second equation, $ax - by = c,$

and we obtain - - - - - - - - $ay + by = an - c.$

Whence - - - - - - - - - $y = \dfrac{an - c}{a + b}.$

By introducing a symbol to represent each of the unknown quantities of the problem, the above solution has the advantage of making known the two required numbers, independently of each other.

What will be the numerical values of x and y, if we suppose

$$n = 48, \quad a = 24, \quad b = 12, \quad \text{and} \quad c = 504.$$

Elimination.

96. The method which has just been explained, of combining two equations, involving two unknown quantities, and deducing therefrom a single equation involving but one, may be extended to three, four, or any number of equations, and is called *Elimination.*

There are three principal methods of elimination:

1st. By addition and subtraction.

2d. By substitution.

3d. By comparison.

We shall discuss these methods separately.

Elimination by Addition and Subtraction.

97. Before considering the case of Elimination, we will explain a new notation which is about to be used.

It often happens, in Algebra, that some of the known quantities of an equation or problem, though entirely independent of each other in regard to their values, have, nevertheless, certain relations which it is desirable to preserve in the discussion. In such case, the second quantity is represented by the same letter, with a small mark over it. Thus, if the first quantity was denoted by a, the second would be denoted by a', and is read, a prime. If there were a third, it would be denoted by a'', and read, a second, &c.

Let us now take the two equations,

$$ax + by = c$$

$$a'x + b'y = c'.$$

If the co-efficients of either of the unknown quantities were the same in both equations; that is, if a were equal to a', or b to b', we might by a simple subtraction form a new equation that would contain but one unknown quantity; and from this equation, the value of that unknown quantity could be deduced.

If, now, both members of the first equation be multiplied by b', the co-efficient of y in the second, and the two members of the second by b, the co-efficient of y in the first, we shall obtain

$$ab'x + bb'y = b'c$$

$$a'bx + bb'y = bc';$$

and by subtracting the second from the first,

$$(ab' - a'b)\,x = b'c - bc';$$

whence,
$$x = \frac{b'c - bc'}{ab' - a'b}.$$

If we multiply the first of the given equations by a', and the second by a, we shall have

$$aa'x + a'by = a'c$$

$$aa'x + ab'y = ac';$$

and by subtracting the second from the first,

$$(a'b - ab')\,y = a'c - ac';$$

whence, $$y = \frac{a'c - ac'}{a'b - ab'};$$

or, if we wish the value for y to have the same denominator with that for x, we change the signs of the numerator and denominator, and write

$$y = \frac{ac' - a'c}{ab' - a'b}$$

The method of elimination just explained, is called the *method by addition and subtraction*, because the unknown quantities disappear by additions and subtractions, after having prepared the equations in such a manner that the same unknown quantity shall have the same co-efficient in both equations.

The formulas $x = \dfrac{b'c - bc'}{ab' - a'b}$, $y = \dfrac{ac' - a'c}{ab' - a'b}$

deduced from the equations

$$ax + by = c$$
$$a'x + b'y = c'$$

will enable us to write the values of x and y immediately, without the trouble of elimination. They contain the germe of a general rule, not before given, for the solution of all similar equations.

RULE.

I. *The first term in the numerator for the value of* x, *is found by beginning at* b′ *and crossing up to* c—*giving* b′c; *the second term is found by crossing from* b *to* c′—*giving* bc′.

II. *For the first term in the numerator of the value for* y, *begin at* a *and cross down to* c′—*giving* ac′; *and for the second term, cross from* a′ *to* c—*giving* a′c.

III. *The first term of the common denominator is found by crossing from* a *to* b′—*giving* ab′; *and the second, by crossing from* a′ *to* b—*giving* a′b.

The manner of obtaining these formulas will be easily remembered, and their applications will be found very simple.

1. What are the values of x and y in the equations,

$$5x + 7y = 43$$
$$11x + 9y = 69.$$

We write immediately,

$$x = \frac{9 \times 43 - \ \ 7 \times 69}{5 \times \ \ 9 - 11 \times \ \ 7} = \frac{387 - 483}{45 - 77} = \frac{-\ \ 96}{-\ \ 32} = 3$$

$$y = \frac{5 \times 69 - 11 \times 43}{-32} = \frac{345 - 473}{-32} = \frac{-128}{-\ \ 32} = 4.$$

2. What are the values of x and y in the equations,

$$3x - \frac{y}{4} = 14$$

$$x - 4y = -11.$$

We write

$$x = \frac{-4 \times 14 - (-\frac{1}{4} \times -11)}{3 \times -4 - 1 \times -\frac{1}{4}} = \frac{-56 - \frac{11}{4}}{-12 + \frac{1}{4}} = 5$$

$$y = \frac{3 \times -11 - 1 \times 14}{-12 + \frac{1}{4}} = \frac{-33 - 14}{-12 + \frac{1}{4}} = 4.$$

Elimination by Substitution.

98. Let us take the two equations

$$5x + 7y = 43 \quad \text{and} \quad 11x + 9y = 69.$$

Find the value of x in the first equation, which gives

$$x = \frac{43 - 7y}{5}.$$

Substitute this value of x in the second equation, and we have

$$11 \times \frac{43 - 7y}{5} + 9y = 69.$$

or $\qquad 473 - 77y + 45y = 345:$

or $\qquad -32y = -128.$

Hence $\qquad y = 4.$

And $\qquad x = \frac{43 - 28}{5} = 3.$

This method, called the method by substitution, consists in finding in one equation the value of one of the unknown quantities, as if the others were already determined, and then substituting this value in the other equations. In this way, new equations are formed from which one of the unknown quantities has been eliminated. We then operate in a similar manner, on the new equations.

Elimination by Comparison.

99. Let us take the two equations,

$$5x + 7y = 43 \quad \text{and} \quad 11x + 9y = 69.$$

Finding the value of x in the first equation, we have

$$x = \frac{43 - 7y}{5}$$

And finding the value of x in the second, we obtain

$$x = \frac{69 - 9y}{11}.$$

Let these two values of x be placed equal to each other, and

we have, $$\frac{43 - 7y}{5} = \frac{69 - 9y}{11};$$

or, $$473 - 77y = 345 - 45y;$$

or, $$-32y = -128.$$

Hence, $$y = 4$$

and, $$x = \frac{69 - 36}{11} = 3.$$

This method of elimination is called the method by comparison, and consists in finding the value of the same unknown quantity in all the equations, and then placing those values equal to each other, two and two. This will give rise to a new set of equations containing one less unknown quantity, and upon which we operate as on the given equations.

The new equations which arise, in the last two methods of elimination, contain fractional terms. This inconvenience is avoided in the first method. The *method by substitution* is, however, advantageously employed whenever the co-efficient of either of the unknown quantities in one of the equations is equal to unity, because then the inconvenience of which we have just spoken does not occur. We shall sometimes have occasion to employ this method, but generally the method by *addition and subtraction* is preferable. When the co-efficients are not too great, we can perform the addition or subtraction at the same time with the multiplication which is made to render the co-efficients of the same unknown quantity equal to each other.

100. Let us now consider the case of three equations involving three unknown quantities.

Take the equations, $$\begin{cases} 5x - 6y + 4z = 15. \\ 7x + 4y - 3z = 19. \\ 2x + \ \ y + 6z = 46. \end{cases}$$

To eliminate z from the first two equations, multiply the first equation by 3 and the second by 4; and since the co-efficients of z have contrary signs, add the two results together: this gives a new equation - - - - - - $43x - 2y = 121$

Multiplying the second equation by 2, a factor of the co-efficient of z in the third equation, and adding them together, we have $16x + 9y = 84$

The question is then reduced to finding the values of x and y, which will satisfy these new equations.

Now, if the first be multiplied by 9, the second by 2, and the results be added together, we find

$$419x = 1257, \text{ whence } x = 3.$$

By means of the two equations involving x and y, we may determine y as we have determined x; but the value of y may be determined more simply, by observing, that by substituting for x its value found above, the last of the two equations becomes,

$$48 + 9y = 84, \quad \text{whence} \quad y = \frac{84 - 48}{9} = 4.$$

In the same manner, by substituting the values of x and y, the first of the three proposed equations becomes,

$$15 - 24 + 4z = 15, \quad \text{whence} \quad z = \frac{24}{4} = 6.$$

101. Hence, if there are m equations involving a like number of unknown quantities, the unknown quantities may be eliminated by the following

RULE.

I. *To eliminate one of the unknown quantities, combine any one of the equations with each of the* m — 1 *others; there will thus be obtained* m — 1 *new equations containing* m — 1 *unknown quantities.*

II. *Eliminate another unknown quantity by combining one of these new equations with the* m — 2 *others; this will give* m — 2 *equations containing* m — 2 *unknown quantities.*

III. *Continue this series of operations until a single equation is obtained containing but one unknown quantity, the value of which can then be found. Then by going back through the series of equations the values of the other unknown quantities may be successively determined.*

102. It often happens that some of the proposed equations do not contain all the unknown quantities. In this case, with a little address, the elimination is very quickly performed.

Take the four equations involving four unknown quantities,

$$\left.\begin{array}{r} 2x - 3y + 2z = 13 \\ 4u - 2x = 30 \end{array}\right\} \begin{array}{l} \cdot\ \cdot\ (1) \\ \cdot\ \cdot\ (2) \end{array} \quad \begin{array}{r} 4y + 2z = 14 \\ 5y + 3u = 32 \end{array} \begin{array}{l} \cdot\ \cdot\ (3). \\ \cdot\ \cdot\ (4). \end{array}$$

By examining these equations, we see that the elimination of z in equations (1) and (3), will give an equation involving x and y; and if we eliminate u in the equations (2) and (4), we shall obtain a second equation, involving x and y. In the first place, the elimination of z, in (1) and (3) gives $7y - 2x = 1$

that of u, in (2) and (4), gives - - - $20y + 6x = 38$

Multiplying the first of these equations by 3, and and adding, we have - - - - $41y = 41$

whence - - - - - - - - $y = 1$

Substituting this value in $7y - 2x = 1$, we find - - - - - - - - - $x = 3$

Substituting for x its value in equation (2), it becomes, $4u - 6 = 30$, whence - - $u = 9$

And substituting for y its value in equation (3), there results - - - - - - $z = 5$

Of indeterminate Problems.

103. In all the preceding reasoning, we have supposed the number of equations equal to the number of unknown quantities. This must be the case in every problem, in order that it may be *determinate;* that is, in order that it may admit of a finite number of solutions.

Let it be required, for example, to find two quantities such, that five times one of them, diminished by three times the other, shall be equal to 12.

If we denote the quantities sought by x and y, we shall have the equation

$$5x - 3y = 12,$$

whence, $$x = \frac{12 + 3y}{5}.$$

Now, by making successively,

$$y = 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \text{ \&c.},$$

there results, $$x = 3, \quad \frac{18}{5}, \quad \frac{21}{5}, \quad \frac{24}{5}, \quad \frac{27}{5} \quad 6, \text{ \&c.},$$

and any two corresponding values of x, y, being substituted in the given equation,

$$5x - 3y = 12$$

will satisfy it equally well: hence, there are an *infinite* number of values for x and y which will satisfy the equation, and consequently, the problem is *indeterminate;* that is, it admits of an infinite number of solutions.

If, however, we impose a second condition, as for example, that the sum of the two quantities shall be equal to 4, we shall have a second equation,

$$x + y = 4;$$

and this, combined with the equation already considered, will give determinate values for x and y.

If we have two equations, involving three unknown quantities, we can eliminate one of the unknown quantities, and thus obtain an equation containing two unknown quantities. This equation, like the preceding, would be satisfied by an infinite number of values, attributed in succession, to the unknown quantities. Since each equation expresses one condition of a problem, therefore, *in order that a problem may be determinate, its enunciation must* contain at least as many *different conditions as there are unknown quantities, and these conditions must be such, that each of them may be expressed by an independent equation; that is, an equation not produced by any combination of the others of the system.*

If, on the contrary, the number of *independent* equations exceeds the number of unknown quantities involved in them, the conditions which they express cannot be fulfilled.

For example, let it be required to find two numbers such that their sum shall be 100, their difference 80, and their product 700.

The equations expressing these conditions are,

$$x + y = 100$$

$$x - y = 80$$

and

$$x \times y = 700.$$

Now, the first two equations determine the values of x and y, viz.,

$$x = 90 \quad \text{and} \quad y = 10.$$

The product of the two numbers is therefore known, and equal to 900. Hence, the third condition cannot be fulfilled.

Had the product been placed equal to 900, all the conditions would have been satisfied, in which case, however, the third would not have been an *independent* equation, since the condition expressed by it, is implied in the other two.

EXAMPLES.

1. Given $2x + 3y = 16$, and $3x - 2y = 11$ to find the values of x and y. *Ans.* $x = 5$, $y = 2$.

2. Given $\frac{2x}{5} + \frac{3y}{4} = \frac{9}{20}$, and $\frac{3x}{4} + \frac{2y}{5} = \frac{61}{120}$ to find the values of x and y. *Ans.* $x = \frac{1}{2}$, $y = \frac{1}{3}$.

3. Given $\frac{x}{7} + 7y = 99$, and $\frac{y}{7} + 7x = 51$ to find the values of x and y. *Ans.* $x = 7$, $y = 14$.

4. Given $\frac{x}{2} - 12 = \frac{y}{4} + 8$, and $\frac{x+y}{5} + \frac{x}{3} - 8 = \frac{2y - x}{4} + 27$ to find the values of x and y. *Ans.* $x = 60$, $y = 40$.

5. Given $\left\{\begin{array}{rl} x + y + z &= 29 \\ x + 2y + 3z &= 62 \\ \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z &= 10 \end{array}\right\}$ to find x, y, and z

Ans. $x = 8$, $y = 9$, $z = 12$

6. Given $\left\{\begin{array}{rrrl} 2x + & 4y - & 3z & = 22 \\ 4x - & 2y + & 5z & = 18 \\ 6x + & 7y - & z & = 63 \end{array}\right\}$ to find x, y, and z.

Ans. $x = 3, \quad y = 7, \quad z = 4.$

7. Given $\left\{\begin{array}{l} x + \frac{1}{2}y + \frac{1}{3}z = 32 \\ \frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z = 15 \\ \frac{1}{4}x + \frac{1}{5}y + \frac{1}{6}z = 12 \end{array}\right\}$ to find x, y, and z.

Ans. $x = 12, \quad y = 20, \quad z = 30.$

8. Given $\left\{\begin{array}{rrrl} 7x - & 2z + & 3u & = 17 \\ 4y - & 2z + & t & = 11 \\ 5y - & 3x - & 2u & = 8 \\ 4y - & 3u + & 2t & = 9 \\ & 3z + & 8u & = 33 \end{array}\right\}$ to find x, y, z, u, and t.

Ans. $x = 2, \quad y = 4, \quad z = 3, \quad u = 3, \quad t = 1.$

QUESTIONS.

1. What fraction is that, to the numerator of which, if 1 be added, its value will be one third, but if one be added to its denominator, its value will be one fourth.

Let the fraction be represented by $\frac{x}{y}$.

Then, by the question $\quad \frac{x+1}{y} = \frac{1}{3} \quad$ and $\quad \frac{x}{y+1} = \frac{1}{4}.$

Whence $\quad 3x + 3 = y, \quad$ and $\quad 4x = y + 1.$

Therefore, by subtracting, $\quad x - 3 = 1 \quad$ or $\quad x = 4;$

and $\quad 3 \times 4 + 3 = 15 = y.$

2. A market woman bought a certain number of eggs at 2 for a penny, and as many more, at 3 for a penny, and having sold them again altogether, at the rate of 5 for $2d.$, found that she had lost $4d.$: how many eggs had she?

Let $2x =$ the whole number of eggs;

then $x =$ the number of eggs of each sort;

and $\frac{1}{2}x =$ the cost of the first sort;

and $\frac{1}{3}x =$ the cost of the second sort;

But $$5 : 2 :: 2x : \frac{4x}{5};$$

hence, $\frac{4x}{5}$ the amount for which the eggs were sold.

Hence, by the question,

$$\frac{1}{2}x + \frac{1}{3}x - \frac{4x}{5} = 4;$$

therefore $$15x + 10x - 24x = 120.$$

Or, $x = 120$ the number of eggs of each sort.

3. A person possessed a capital of 30,000 dollars, for which he drew a certain interest per annum; but he owed the sum of 20,000 dollars, for which he paid a certain interest. The interest that he received exceeded that which he paid by 800 dollars. Another person possessed $35,000, for which he received interest at the second of the above rates; but he owed 24,000 dollars, for which he paid interest at the first of the above rates. The interest that he received exceeded that which he paid by 310 dollars. Required the two rates of interest.

Let x and y denote the two rates of interest: that is, the interest of $100 for one year.

To obtain the interest of $30,000 at the first rate, denoted by x, we form the proportion

$$100 : x :: 30{,}000 :: \frac{30{,}000x}{100} \text{ or } 300x.$$

And for the interest $20,000, the rate being y,

$$100 : y :: 20{,}000 :: \frac{20{,}000y}{100} \text{ or } 200y.$$

But from the enunciation, the difference between these two interests is equal to 800 dollars.

We have, then, for the first equation of the problem,

$$300x - 200y = 800.$$

By expressing the second condition of the problem algebraically, we obtain the other equation,

$$350y - 240x = 310.$$

Both members of the first equation being divisible by 100, and those of the second by 10, we may put the following, in place of them:

$$3x - 2y = 8, \qquad 35y - 24x = 31.$$

To eliminate x, multiply the first equation by 8, and then add it to the second; there results

$$19y = 95, \quad \text{whence} \quad y = 5.$$

Substituting for y its value in the first equation, this equation becomes

$$3x - 10 = 8, \quad \text{whence} \quad x = 6;$$

Therefore, the first rate is 6 per cent., and the second 5.

Verification.

\$30,000,	placed at	6 per cent.,	gives	$300 \times 6 = \$1800.$
\$20,000	do.	5	do.	$200 \times 5 = \$1000.$

And we have $\qquad 1800 - 1000 = 800.$

The second condition can be verified in the same manner.

4. There are three ingots formed by mixing together three metals in different proportions.

One pound of the first contains 7 ounces of silver, 3 ounces of copper, and 6 ounces of pewter.

One pound of the second contains 12 ounces of silver, 3 ounces of copper, and 1 ounce of pewter.

One pound of the third contains 4 ounces of silver, 7 ounces of copper, and 5 ounces of pewter.

It is required to form from these three, 1 pound of a fourth ingot which shall contain 8 ounces of silver, $3\frac{3}{4}$ ounces of copper, and $4\frac{1}{4}$ ounces of pewter.

Let $x =$ the number of ounces taken from the first.

$y =$ the number of ounces taken from the second.

$z =$ the number of ounces taken from the third.

Now, since 1 pound or 16 ounces of the first ingot contains 7 ounces of silver, one ounce will contain $\frac{1}{16}$ of 7 ounces: that is, $\frac{7}{16}$ ounces; and

x ounces will contain $\frac{7x}{16}$ ounces of silver.

y ounces will contain $\frac{12y}{16}$ ounces of silver.

z ounces will contain $\frac{4z}{16}$ ounces of silver.

But since 1 pound of the new ingot is to contain 8 ounces of silver, we have

$$\frac{7x}{16} + \frac{12y}{16} + \frac{4z}{16} = 8;$$

or, reducing to entire terms,

$$7x + 12y + 4z = 128.$$

For the copper, $3x + 3y + 7z = 60$;

and for the pewter, $6x + y + 5z = 68$.

As the co-efficients of y in these three equations, are the most simple, we will eliminate this unknown quantity first.

Multiplying the second equation by 4 and subtracting the first, gives

$$5x + 24z = 112.$$

Multiplying the third equation by 3 and subtracting the second, gives

$$15x + 8z = 144.$$

Multiplying the last equation by 3 and subtracting the first, gives

$$40x = 320,$$

whence $x = 8$

Substituting this value of x in the equation

$$5x + 24z = 112,$$

it becomes $40 + 24z = 112$, whence $z = 3$.

Lastly, the two values $x = 8$ and $z = 3$, being substituted in the equation

$$6x + y + 5z = 68$$

give $48 + y + 15 = 68$, whence $y = 5$.

Therefore, in order to form a pound of the fourth ingot, we must take 8 ounces of the first, 5 ounces of the second, and 3 of the third.

Verification.

If there be 7 ounces of silver in 16 ounces of the first ingot, in 8 ounces of it, there should be a number of ounces of silver expressed by $\frac{7 \times 8}{16}$.

In like manner, $\frac{12 \times 5}{16}$ and $\frac{4 \times 3}{16}$ will express the quantity of silver contained in 5 ounces of the second ingot, and 3 ounces of the third. Now, we have

$$\frac{7 \times 8}{16} + \frac{12 \times 5}{16} + \frac{4 \times 3}{16} = \frac{128}{16} = 8;$$

therefore, a pound of the fourth ingot contains 8 ounces of silver, as required by the enunciation. The same conditions may be verified relative to the copper and pewter.

5. What two numbers are those, whose difference is 7, and sum 33? *Ans.* 13 and 20.

6. To divide the number 75 into two such parts, that three times the greater may exceed seven times the less by 15.
Ans. 54 and 21.

7. In a mixture of wine and cider, $\frac{1}{2}$ of the whole plus 25 gallons was wine, and $\frac{1}{3}$ part minus 5 gallons was cider; how many gallons were there of each?
Ans. 85 of wine, and 35 of cider.

8. A bill of £120 was paid in guineas and moidores, and the number of pieces of both sorts that were used was just 100; if the guinea were estimated at 21*s*., and the moidore at 27*s*., how many were there of each? *Ans.* 50 of each.

9. Two travellers set out at the same time from London and York, whose distance apart is 150 miles; one of them goes 8 miles a day, and the other 7; in what time will they meet?
Ans. In 10 days.

10. At a certain election, 375 persons voted for two candidates, and the candidate chosen had a majority of 91; how many voted for each? *Ans.* 233 for one, and 142 for the other.

11. A's age is double of B's, and B's is triple of C's, and the sum of all their ages is 140; what is the age of each?

Ans. A's $= 84$, B's $= 42$, and C's $= 14$.

12 A person bought a chaise, horse, and harness, for £60; the horse came to twice the price of the harness, and the chaise to twice the price of the horse and harness; what did he give for each?

Ans. £13 6*s.* 8*d.* for the horse.
£ 6 13*s.* 4*d.* for the harness.
£40 for the chaise.

13. Two persons, A and B, have both the same income. A saves $\frac{1}{5}$ of his yearly; but B, by spending £50 per annum more than A, at the end of 4 years finds himself £100 in debt; what is the income of each? *Ans.* £125.

14. A person has two horses, and a saddle worth £50; now, if the saddle be put on the back of the first horse, it will make his value double that of the second; but if it be put on the back of the second, it will make his value triple that of the first; what is the value of each horse?

Ans. One £30, and the other £40.

15. To divide the number 36 into three such parts, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, may be all equal to each other. *Ans.* 8, 12, and 16.

16. A footman agreed to serve his master for £8 a year and a livery, but was turned away at the end of 7 months, and received only £2 13*s.* 4*d.* and his livery; what was its value?

Ans. £4 16*s.*

17. To divide the number 90 into four such parts, that if the first be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the sum, difference, product, and quotient so obtained, will be all equal to each other.

Ans. The parts are 18, 22, 10, and 40.

18. The hour and minute hands of a clock are exactly together at 12 o'clock; when are they next together?

Ans. 1 h. $5\frac{5}{11}$ min.

19. A man and his wife usually drank out a cask of beer in 12 days; but when the man was from home, it lasted the woman 30 days; how many days would the man be in drinking it alone?

Ans. 20 days.

20. If A and B together can perform a piece of work in 8 days, A and C together in 9 days, and B and C in 10 days; how many days would it take each person to perform the same work alone? *Ans.* A $14\frac{34}{49}$ days, B $17\frac{23}{41}$, and C $23\frac{7}{31}$.

21. A laborer can do a certain work expressed by a, in a time expressed by b; a second laborer, the work c in a time d; a third, the work e in a time f. Required the time it would take the three laborers, working together, to perform the work g.

$$Ans.\ x = \frac{bdfg}{adf + bcf + bde}.$$

Application.

$a = 27;\ b = 4 \mid c = 35;\ d = 6 \mid e = 40;\ f = 12 \mid g = 191;$

x will be found equal to 12.

22. If 32 pounds of sea water contain 1 pound of salt, how much fresh water must be added to these 32 pounds, in order that the quantity of salt contained in 32 pounds of the new mixture shall be reduced to 2 ounces, or $\frac{1}{8}$ of a pound?

Ans. 224 lbs.

23. A number is expressed by three figures; the sum of these figures is 11; the figure in the place of units is double that in the place of hundreds; and when 297 is added to this number, the sum obtained is expressed by the figures of this number reversed. What is the number? *Ans.* 326.

24. A person who possessed 100,000 dollars, placed the greater part of it out at 5 per cent. interest, and the other part at 4 per cent. The interest which he received for the whole amounted to 4640 dollars. Required the two parts.

Ans. \$64,000 and \$36,000.

25. A person possessed a certain capital, which he placed out at a certain interest. Another person possessed 10,000 dollars more than the first, and putting out his capital 1 per cent. more advantageously, had an income greater by 800 dollars. A third, possessed 15,000 dollars more than the first, and putting out his capital 2 per cent. more advantageously, had an income greater by 1500 dollars. Required the capitals, and the three rates of interest.

Sums at interest,	\$30,000,	\$40,000,	\$45,000.	
Rates of interest,	4	5	6	per cent

26. A banker has two kinds of money; it takes a pieces of the first to make a crown, and b of the second to make the same sum. Some one offers him a crown for c pieces. How many of each kind must the banker give him?

$$\textit{Ans. } \text{1st kind, } \frac{a(c-b)}{a-b}; \text{ 2d kind, } \frac{b(a-c)}{a-b}.$$

27. Find what each of three persons, A, B, C, is worth, knowing, 1st, that what A is worth added to l times what B and C are worth, is equal to p; 2d, that what B is worth added to m times what A and C are worth, is equal to q; 3d, that what C is worth added to n times what A and B are worth, is equal to r.

If we denote by x what A, B, and C, are worth, we introduce into the *calculus* an auxiliary unknown quantity, and resolve the question in a very simple manner. The term *calculus,* in its general sense, *denotes any operation performed on algebraic quantities.*

28. Find the values of the estates of six persons, A, B, C, D, E, F, from the following conditions: 1st. The sum of the estates of A and B is equal to a; that of C and D is equal to b; and that of E and F is equal to c. 2d. The estate of A is worth m times that of C; the estate of D is worth n times that of E, and the estate of F is worth p times that of B.

This problem may be resolved by means of a single equation, involving but one unknown quantity.

Explanation of Negative Results.

104. The algebraic signs are an abbreviated language. They indicate certain operations which are to be performed on the quantities before which they are placed.

The operation indicated by a particular sign, must be performed on every quantity before which the sign is placed. Indeed, the principles of Algebra are all established upon the supposition, that each particular sign which is employed means always the same thing; and that whatever it requires is strictly performed. Thus, if the sign of a quantity is $+$, we understand that the quantity is to be added; if the sign is $-$, we understand that it is to be subtracted.

For example, if we have -4, it indicates that this 4 is to be subtracted from some other number, or that it is the result of a subtraction but partially made.

If it were required to subtract 20 from 16, the subtraction could not be made by the rules of arithmetic, since 20 is greater than 16. By observing that

$$20 = 16 + 4,$$

we may express the subtraction thus,

$$16 - 20 = 16 - 16 - 4 = -4.$$

We thus make the subtraction of 20 from 16 as far as it is possible, and obtain a remainder 4 with a minus sign, which indicates that 4 is still to be treated as a subtractive quantity.

To show the necessity of giving to this remainder its proper sign, let us suppose that 10 is to be added to the difference of $16 - 20$; or what is the same thing, that 20 is to be subtracted from 26.

The numbers would then be written

$$\begin{array}{rcr} 16 - 20 & = & -\ 4 \\ +\ 10 & = & +\ 10 \\ \hline 26 - 20 & = & +\ \ 6 \end{array};$$

and had the $-$ sign not been preserved in the first subtraction, the second result would have been $+14$ instead of $+6$.

105. If the sum of the negative quantities in the first member of the equation, exceeds the sum of the positive quantities, the second member of the equation will be negative, and the verification of the equation will show it to be so.

For example, if $\qquad a - b = c,$

and we make $\quad a = 15$ and $b = 18$, c will be $= -3$.

Now, the essential sign of c is different from its algebraic sign in the equation. This arises from the circumstance, that the equation

$$a - b = c$$

expresses *generally*, the difference between a and b, without indicating which of them is the greater. When, therefore, we attribute particular values to a and b, the *sign* of c, as well as its value, becomes known.

We will illustrate these remarks by a few examples.

1. To find a number, which added to the number b, will give a sum equal to the number a.

Let $x =$ the required number.

Then, by the conditions

$$x + b = a, \text{ whence } x = a - b.$$

This expression, or *formula*, will give the algebraic value of x in all the particular cases of this problem.

For example, let $a = 47$ and $b = 29$;

then, $x = 47 - 29 = 18$.

Again, let $a = 24$ and $b = 31$;

then, $x = 24 - 31 = -7$.

This last value of x, is called a *negative solution*. How is it to be interpreted?

If we consider it as a purely arithmetical result, that is, as arising from a series of operations in which all the quantities are regarded as positive, and in which the terms *add* and *subtract* imply, respectively, augmentation and diminution, the problem will obviously be impossible for the last values attributed to a and b; for, the number b is already greater than 24.

Considered, however, algebraically, it is not so; for we have found the value of x to be -7, and this number added, in the algebraic sense, to 31, gives 24 for the algebraic sum, and therefore satisfies both the equation and enunciation.

2. A father has lived a number a of years, his son a number of years expressed by b. Find in how many years the age of the son will be one fourth the age of the father.

Let $x =$ the required number of years.

Then $a + x =$ the age of the father } at the end of the re-
and $b + x =$ the age of the son } quired time.

Hence, by the question

$$\frac{a+x}{4} = b + x; \text{ whence, } x = \frac{a-4b}{3}.$$

Suppose $a = 54$, and $b = 9$; then $x = \dfrac{54 - 36}{3} = \dfrac{18}{3} = 6$.

The father being 54 years old, and the son 9, in 6 years the father will be 60 years old, and his son 15; now 15 is the fourth of 60; hence, $x = 6$ satisfies the enunciation.

Let us now suppose $a = 45$, and $b = 15$;

then, $$x = \frac{45 - 60}{3} = -5.$$

If we substitute this value of x in the equation of condition,

$$\frac{a + x}{4} = b + x,$$

we obtain, $$\frac{45 - 5}{4} = 15 - 5;$$

and $$10 = 10.$$

Hence, -5 substituted for x, verifies the equation, and therefore is a true answer.

Now, the positive result which was obtained, shows that the age of the father will be four times that of the son at the expiration of 6 years from the time when their ages were considered; while the negative result, indicates that the age of the father was four times that of his son, 5 years *previous* to the time when their ages were compared.

The question, taken in its general, or algebraic sense, demands *the time*, at which the age of the father was four times that of the son. In stating it, we supposed that the age of the father was to be augmented; and so it was, by the first supposition. But the conditions imposed by the second supposition, required the age of the father to be diminished, and the algebraic result conformed to this condition, by appearing with a negative sign. If we wished the result, under the second supposition, to have a positive sign, we might alter the enunciation by demanding, *how many years since the age of the father was four times that of the son.*

If $x =$ the number of years, we shall have

$$\frac{a - x}{4} = b - x: \text{ hence, } x = \frac{4b - a}{3}.$$

If $a = 45$ and $b = 15$, x will be equal to 5.

Reasoning from analogy, we establish the following general principles,

1st. *Every negative value found for the unknown quantity in a problem of the first degree, will, when taken with its proper sign, verify the equation from which it was derived.*

2d. *That this negative value, taken with its proper sign, will also satisfy the enunciation of the problem, understood in its algebraic sense.*

3d. *The negative result shows that the enunciation is impossible, regarded in its arithmetical sense. The language of Algebra detects the error of the arithmetical enunciation, and indicates the general relation of the quantities.*

4th. *The negative result, considered without reference to its sign, may be regarded as the answer to a problem of which the enunciation only differs from the one proposed in this: that certain quantities which were additive have become subtractive, and reciprocally.*

106. As a further illustration of the change which an algebraic sign may produce in the enunciation of a problem, let us resume that of the laborer (page 76).

Under the supposition that the laborer receives a sum c, we have the equations

$$\left.\begin{aligned} x + \ \ y &= n \\ ax - by &= c \end{aligned}\right\} \quad \text{whence,} \quad x = \frac{bn + c}{a + b}, \quad y = \frac{an - c}{a + b}.$$

If at the end of the time, the laborer, instead of receiving a sum c, owed for his board a sum equal to c, then, by would be greater than ax, and under this supposition, we should have the equations

$$x + y = n \quad \text{and} \quad ax - by = -c.$$

Now, it is plain that we can obtain immediately the values of x and y, in the last equations, by merely changing the sign of c in each of the values found from the equations above; this gives

$$x = \frac{bn - c}{a + b}, \quad y = \frac{an + c}{a + b}.$$

The results for both enunciations, may be comprehended in the same formulas, by writing

$$x = \frac{bn \pm c}{a + b}; \qquad y = \frac{an \mp c}{a + b}.$$

The double sign $\pm$, is read *plus* or *minus*, and $\mp$, is read, *minus* or *plus*. The upper signs correspond to the case in which

the laborer received, and the lower signs, to the case in which he owed a sum c. These formulas also comprehend the case in which, in a settlement between the laborer and his employer, their accounts balance. This supposes $c = 0$, which gives

$$x = \frac{bn}{a+b}; \qquad y = \frac{an}{a+b}.$$

Discussion of Problems. Explanation of the terms Nothing and Infinity.

107. When a problem has been resolved generally, that is, by means of letters and signs, it is often required to determine what the values of the unknown quantities become, when particular suppositions are made upon the quantities which are given. The determination of these values, and the interpretation of the peculiar results obtained, form what is called the *discussion of the problem.*

The discussion of the following question presents nearly all the circumstances which are met with in problems of the first degree.

108. Two couriers are travelling along the same right line and in the same direction from R′ toward R. The number of miles travelled by one of them per hour is expressed by m, and the number of miles travelled by the other per hour, is expressed by n. Now, at a given time, say 12 o'clock, the distance between them is equal to a number of miles expressed by a: required the time when they will be together.

R′ ...... A ———— B ———— R.

At 12 o'clock suppose the forward courier to be at B, the other at A, and R to be the point at which they will be together.

Then, AB $= a$, their distance apart at 12 o'clock.

Let $t =$ the number of hours which must elapse, before they come together;

and $x =$ the distance BR, which is to be passed over by the forward courier.

Then, since the rate per hour, multiplied by the number of hours, will give the distance passed over by each, we have,

$$t \times m = a + x = \text{AR}$$
$$t \times n = x \qquad = \text{BR}.$$

Hence by subtracting,

$$t(m-n)=a,$$

and hence, $$t=\frac{a}{m-n}.$$

Now, so long as $m>n$, t will be positive, and the problem will be solved in the arithmetical sense of the enunciation. For, if $m>n$, the courier from A will travel faster than the courier from B, and will therefore be continually gaining on him: the interval which separates them will diminish more and more, until it becomes 0, and then the couriers will be found upon the same point of the line.

In this case, the time t, which elapses, must be added to 12 o'clock, to obtain the time when they are together.

But, if we suppose $m<n$, then, $m-n$ will be negative, and the value of t will be negative. How is this result to be interpreted?

It is easily explained from the nature of the question, which considered in its most general sense, demands *the time when the couriers are together.*

Now, under the second supposition, the courier which is in advance, travels the fastest, and therefore will continue to separate himself from the other courier. At 12 o'clock the distance between them was equal to a: after 12 o'clock it is greater than a; and as the rate of travel has not been changed, it follows that previous to 12 o'clock the distance must have been less than a. At a certain hour, therefore, before 12, the distance between them must have been equal to nothing, or the couriers were together at some point R′. The precise hour is found by *subtracting* the value of t from 12 o'clock.

This example, therefore, conforms to the general principle, that, *if the conditions of a problem are such as to render the unknown quantity essentially negative, it will appear in the result with the minus sign, whenever it has been regarded as positive in the enunciation.*

If we wish to find the distances AR and BR, passed over by the two couriers before coming together, we may take the equation

$$t=\frac{a}{m-n}$$

and multiply both members by the rates of travel respectively: this will give

$$AR = mt = \frac{ma}{m-n} \quad \text{and}$$

$$BR = nt = \frac{na}{m-n}.$$

Also, $$AR' = -mt = \frac{ma}{m-n}$$

and $$BR' = -nt = \frac{na}{m-n};$$

from which we see, that the two distances AR and BR, will both be positive when estimated toward the right, and that AR′ and BR′ will both be negative when estimated in the contrary direction.

109. To explain the terms nothing and infinity, let us consider the equation

$$t = \frac{a}{m-n}.$$

If the couriers travel at different rates, $m - n$ will be a finite quantity, and its sign will depend on the relative values of m and n. Designate this quantity by A.

Now, if we suppose $a = 0$, we shall have

$$t = \frac{0}{A}; \quad \text{or} \quad t \times A = 0;$$

an equation which can only be satisfied by making $t = 0$.

To interpret this result, let us go back to the enunciation of the problem. If $a = 0$, the couriers are together at 12 o'clock; and since they travel at different rates, they can never be again together: hence, t can have no other value than 0. Therefore, we conclude that, *the quotient of* 0 *divided by a finite quantity, is* 0.

110. Let us resume the equation

$$t = \frac{a}{m-n}.$$

If in this equation we make $m = n$, then $m - n = 0$, and the value of t will reduce to

$$t = \frac{a}{0} \quad \text{or} \quad t \times 0 = a;$$

an equation which cannot be satisfied for any finite value of t.

In order to interpret this new result, let us go back to the enunciation of the question. We see at once, that it is absolutely impossible to satisfy the enunciation for any finite value for t; for, whatever time we allow to the two couriers, they can never come together, since being once separated by an interval a, and travelling equally fast, this interval will always be preserved.

Hence, the result, $\frac{a}{0}$ may be regarded as a sign of impossibility for any *finite* value of t.

Nevertheless, algebraists consider the result,

$$t = \frac{a}{0},$$

as forming a species of value, to which they have given the name of *infinite value*, for this reason:

When the difference $m - n$, without being absolutely nothing, is supposed to be very small, the result

$$t = \frac{a}{m - n}.$$

is very great.

Take, for example, $\quad m - n = 0,01.$

Then $$t = \frac{a}{m - n} = \frac{a}{0.01} = 100a.$$

Again, take $\quad m - n = 0.001$, and we have

$$\frac{a}{m - n} = \frac{a}{0.001} = 1000a.$$

In short, if the difference between the rates is not zero, the couriers will come together at some point of the line, and the time will become greater and greater, as this difference is diminished.

Hence, from analogy, *if the difference between the rates is less than any assignable number, the time expressed by*

$$t = \frac{a}{m - n} = \frac{a}{0},$$

will be greater than any assignable or finite number. Therefore for brevity, we say, when $m - n = 0$, the result,

$$t = \frac{a}{m - n} = \frac{a}{0}.$$

becomes equal to *infinity*, which we designate by the character ∞.

Hence we conclude, that *a finite quantity divided by* 0, *gives a quotient greater than any assignable quantity, which we call,* INFINITY.

111. Again, let A represent any finite number: then, since the value of a fraction increases as its numerator becomes greater with reference to its denominator, the expression

$$\frac{A}{0},$$

is a proper symbol to represent an *infinite* quantity; that is, a quantity greater than any assignable quantity.

Since the value of a fraction diminishes as its denominator becomes greater with reference to its numerator, the expression

$$\frac{A}{\infty}$$

is a proper symbol for a quantity less than any assignable quantity. Hence,

$$\frac{A}{0} \text{ and } \infty$$

are synonymous symbols; and so likewise, are

$$\frac{A}{\infty} \text{ and } 0.$$

We have been thus particular in explaining these ideas of infinity, because there are some questions of such a nature, that infinity may be considered as the true answer to the enunciation.

In the case just considered, where $m = n$, it will be perceived that there is not, properly speaking, any solution in *finite and determinate numbers*; but the value of the unknown quantity is found to be infinite.

112. If, in addition to the hypothesis $m = n$, we also suppose that $a = 0$, we have $t = \frac{0}{0}$, or $t \times 0 = 0$; a result which will be satisfied by any value of t.

To interpret this result, let us consider again the enunciation, from which it is perceived, that if the two couriers travel equally fast, and are once at the same point, they ought, ever after, to be together, and consequently the required time is entirely unde-

termined. Therefore, the expression $\frac{0}{0}$ is, in this case, the symbol of an *indeterminate quantity*.

The preceding suppositions are the only ones that lead to remarkable results; and they are sufficient to show to beginners the manner in which the results of Algebra answer to all the circumstances of the enunciation of a problem.

113. It should be observed, that the expression $\frac{0}{0}$ is not a certain symbol of *indetermination*, but frequently arises from *the existence of a common factor* in each term of the fraction, which factor becomes nothing, in consequence of a particular hypothesis.

For example, suppose the value of the unknown quantity to be

$$x = \frac{a^3 - b^3}{a^2 - b^2}.$$

If, in this formula, a is made equal to b, there results

$$x = \frac{0}{0}$$

But observe (Art. 48), that

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

and $$a^2 - b^2 = (a - b)(a + b),$$

hence, we have

$$x = \frac{(a - b)(a^2 + ab + b^2)}{(a - b)(a + b)}$$

Now, if we suppress the common factor $a - b$, and then suppose $a = b$, we shall have

$$x = \frac{a^2 + ab + b^2}{a + b} = \frac{3a^2}{2a} = \frac{3a}{2}.$$

Let us suppose, that in another example, we have

$$x = \frac{a^2 - b^2}{(a - b)^2}.$$

If we suppose $a = b$, we have

$$x = \frac{0}{0}.$$

If, however, we suppress the factor common to the numerator and denominator, in the value of x, we have,

$$x = \frac{(a + b)(a - b)}{(a - b)(a - b)} = \frac{a + b}{a - b} = \frac{2b}{0} = \infty.$$

Therefore we conclude, that before pronouncing upon the true value of the fraction,

$$\frac{0}{0}$$

it is necessary to ascertain whether the two terms do not contain a common factor. If they do not, we conclude that the fraction is really *indeterminate*. If they do contain one, suppress it, and then make the particular hypothesis; this will give the true value of the fraction, which will assume one of the three forms

$$\frac{A}{B}, \quad \frac{A}{0}, \quad \frac{0}{0},$$

that is, it will be *determinate*, *infinite*, or *indeterminate*.

This observation is very useful in the discussion of problems.

Of Inequalities.

114. In the discussion of problems, we have often occasion to suppose quantities *unequal*, and to perform transformations upon them, analogous to those executed upon *equalities*. We sometimes do this, to establish the necessary relations between the given quantities, in order that the problem may be susceptible of a direct, or at least, of a real solution. We often do it, to fix the limits between which the particular values of certain given quantities must be found, in order that the enunciation may fulfil a particular condition. Now, although the principles established for equations are, in general, applicable to inequalities, there are nevertheless some exceptions, of which it is necessary to speak, in order to put the beginner upon his guard against some errors that he might commit, in making use of the sign of inequality. These exceptions arise from the introduction of *negative expressions* into the calculus, as *quantities*.

In order to be clearly understood, we will give examples of the different transformations to which inequalities may be subjected, taking care to point out the exceptions to which these transformations are liable.

115. Two inequalities are said to subsist in the same sense, when the greater quantity stands at the left in both, or at the right in both; and in a contrary sense, when the greater quantity stands at the right in one, and at the left in the other.

Thus, $25 > 20$ and $18 > 10$, or $6 < 8$ and $7 < 9$,

are inequalities which subsist in the same sense; and the inequalities

$$15 > 13 \text{ and } 12 < 14,$$

subsist in a contrary sense.

1. *If we add the same quantity to both members of an inequality, or subtract the same quantity from both members, the resulting inequality will subsist in the same sense.*

Thus, take $8 > 6$; by adding 5, we still have

$$8 + 5 > 6 + 5;$$

and subtracting 5, we have

$$8 - 5 > 6 - 5.$$

When the two members of an inequality are both negative, that one is the least, algebraically considered, which contains the greatest number of units. Thus, $-25 < -20$; and if 30 be added to both members, we have $5 < 10$. This must be understood entirely in an algebraic sense, and arises from the convention before established, to consider all quantities preceded by the minus sign, as subtractive.

The principle first enunciated, serves to transpose certain terms from one member of the inequality to the other. Take, for example, the inequality

$$a^2 + b^2 > 3b^2 - 2a^2;$$

there will result, by transposing,

$$a^2 + 2a^2 > 3b^2 - b^2, \text{ or } 3a^2 > 2b^2.$$

2. *If two inequalities subsist in the same sense, and we add them member to member, the resulting inequality will also subsist in the same sense.*

Thus, add $\quad a > b, \ c > d, \ e > f$: and

there results $\quad a + c + e > b + d + f.$

But this is not always the case, when we subtract, member from member, two inequalities established in the same sense.

Let there be the two inequalities $4 < 7$ and $2 < 3$, we have

$$4 - 2 \text{ or } 2 < 7 - 3 \text{ or } 4.$$

But if we have the inequalities $9 < 10$ and $6 < 8$, by subtracting we have

$$9 - 6 \text{ or } 3 > 10 - 8 \text{ or } 2.$$

We should then avoid this transformation as much as possible, or if we employ it, determine in which sense the resulting inequality exists.

3. *If the two members of an inequality be multiplied by a positive number, the resulting inequality will exist in the same sense.*

Thus, $a < b$, will give $3a < 3b$;

and, $-a < -b$, $-3a < -3b$.

This principle serves to make the denominators disappear.

From the inequality $\frac{a^2 - b^2}{2d} > \frac{c^2 - d^2}{3a}$, we deduce, by multiplying by $6ad$,

$$3a(a^2 - b^2) > 2d(c^2 - d^2),$$

and the same principle is true for division.

But, *when the two members of an inequality are multiplied or divided by a negative number, the inequality will subsist in a contrary sense.*

Take, for example, $8 > 7$; multiplying by -3, we have

$$-24 < -21.$$

In like manner, $8 > 7$ gives $\frac{8}{-3}$, or $-\frac{8}{3} < -\frac{7}{3}$.

Therefore, when the two members of an inequality are multiplied or divided by a number expressed algebraically, it is necessary to ascertain whether the *multiplier* or *divisor* is negative; for, in that case, the inequality will exist in a contrary sense.

4. *It is not permitted to change the signs of the two members of an inequality, unless we establish the resulting inequality in a contrary sense;* for this transformation is evidently the same as multiplying the two members by -1.

5. *Both members of an inequality between positive numbers can be squared, and the inequality will exist in the same sense.*

Thus, from $5 > 3$, we deduce, $25 > 9$; from $a + b > c$, we find

$$(a + b)^2 > c^2.$$

6. *When the signs of both members of the inequality are not known, we cannot tell before the operation is performed, in which sense the resulting inequality will exist.*

For example, $-2 < 3$ gives $(-2)^2$ or $4 < 9$; but $3 > -5$ gives, on the contrary, $(3)^2$ or $9 < (-5)^2$ or 25.

We must, then, before squaring, ascertain the signs of the two members.

EXAMPLES.

1. Find the limit of the value of x in the expression

$$5x - 6 > 19. \qquad Ans.\ x > 5.$$

2. Find the limit of the value of x in the expression

$$3x + \frac{14}{2}x - 30 > 10. \qquad Ans.\ x > 4.$$

3. Find the limit of the value of x in the expression

$$\frac{1}{6}x - \frac{1}{3}x + \frac{x}{2} + \frac{13}{2} > \frac{17}{2}. \qquad Ans.\ x > 6.$$

4. Find the limit of the value of x in the inequalities

$$\frac{ax}{5} + bx - ab > \frac{a^2}{5}.$$

$$\frac{bx}{7} - ax + ab < \frac{b^2}{7}.$$

5. The double of a number diminished by 5 is greater than 25, and triple the number diminished by 7, is less than double the number increased by 13. Required a number which shall satisfy the conditions.

By the question, we have

$$2x - 5 > 25.$$

$$3x - 7 < 2x + 13.$$

Resolving these inequalities, we have $x > 15$ and $x < 20$. Any number, therefore, either entire or fractional, comprised between 15 and 20, will satisfy the conditions.

CHAPTER V.

EXTRACTION OF THE SQUARE ROOT OF NUMBERS.—FORMATION OF THE SQUARE AND EXTRACTION OF THE SQUARE ROOT OF ALGEBRAIC QUANTITIES.—CALCULUS OF RADICALS OF THE SECOND DEGREE.

116. THE *square* or second power of a number, is the product which arises from multiplying that number by itself once: for example, 49 is the square of 7, and 144 is the square of 12.

The *square root* of a number, is that number which multiplied by itself once will produce the given number. Thus, 7 is the square root of 49, and 12 the square root of 144: for, $7 \times 7 = 49$, and $12 \times 12 = 144$.

The square of a number, either entire or fractional, is easily found, being always obtained by multiplying the number by itself once. The extraction of the square root is, however, attended with some difficulty, and requires particular explanation.

The first ten numbers are,

1, 2, 3, 4, 5, 6, 7, 8, 9, 10,

and their squares,

1, 4, 9, 16, 25, 36, 49, 64, 81, 100:

and reciprocally, the numbers of the first line are the square roots of the corresponding numbers of the second. We may also remark that, *the square of a number expressed by a single figure, will contain no figure of a higher denomination than tens.*

The numbers of the last line, 1, 4, 9, 16, &c., and all other numbers which can be produced by the multiplication of a number by itself, are called *perfect squares.*

It is obvious, that there are but nine perfect squares among all the numbers which can be expressed by one or two figures: the square roots of all other numbers expressed by one or two figures

will be found between two whole numbers differing from each other by unity. Thus, the square root of 55, comprised between the perfect squares 49 and 64, is greater than 7 and less than 8. Also, the square root of 91, comprised between the perfect squares 81 and 100, is greater than 9 and less than 10.

Every number may be regarded as made up of a certain number of tens and a certain number of units. Thus 64 is made up of 6 tens and 4 units, and may be expressed under the form

$$60 + 4 = 64.$$

Now, if we represent the tens by a and the units by b, we shall have

$$a + b \quad = 60 + 4,$$

and
$$(a + b)^2 = (60 + 4)^2,$$

and consequently,

$$a^2 + 2ab + b^2 \; = (60)^2 + 2 \times 60 \times 4 + (4)^2 = 4096.$$

Hence, the square of a number composed of tens and units contains, *the square of the tens, plus twice the product of the tens by the units, plus the square of the units.*

117. If now, we make the units 1, 2, 3, 4, &c., tens, by annexing to each a cipher, we shall have,

10, 20, 30, 40, 50, 60, 70, 80, 90, 100;

and for their squares,

100, 400, 900, 1600, 2500, 3600, 4900, 6400, 8100, 10000;

from which we see that the square of one ten is 100, the square of two tens, 400, &c.: and hence, *the square of tens will contain no figure of a less denomination than hundreds, nor of a higher name than thousands.*

Let us now take any number, as 78, and square it. We have

$$78 = 70 + 8;$$

that is, equal to 7 tens, or 70, plus 8 units.

Seven tens, or 70 squared - -	$(70)^2 = 4900$
twice the tens by the units is,	$2 \times 70 \times 8 = 1120$
square of the units is, - - -	$(8)^2 = 64$
hence, - - - - - -	$(78)^2 = 6084.$

Let us now reverse this process and find the square root of 6084.

Since this number is composed of more than two places of figures, its roots will contain more than one. But since it is less than 10000, which is the square of 100, the root will contain but two figures; that is, units and tens.

$$6\dot{0}\,8\dot{4}$$

Now, the square of the tens must be found in the two left-hand figures which we will separate from the other two, by placing a point over the the place of units, and another over the place of hundreds. These parts, of two figures each, are called *periods*. The part 60 is comprised between the two squares 49 and 64, of which the roots are 7 and 8: hence, 7 *is the figure of the tens sought;* and the required root is composed of 7 tens and a certain number of units.

The figure 7 being found, we write it on the right of the given number, from which we separate it by a vertical line: then we subtract its square 49 from 60, which leaves a remainder of 11, to which we bring down the two next figures 84. The result of this operation is 1184, and this number is made up of *twice the product of the tens by the units plus the square of the units.*

$$\begin{array}{r|l}
 & 6\dot{0}\,8\dot{4}\ \vert\ 78 \\
 & 49 \\ \hline
7 \times 2 = 14\,8 & 118\,4 \\
 & \underline{118\,4} \\
 & \quad\ \ 0
\end{array}$$

But since tens multiplied by units cannot give a product of a less name than tens, it follows that the last figure 4 can form no part of the double product of the tens by the units: this double product is therefore found in the part 118.

Now, if we double the tens, which gives 14, and then divide 118 by 14, the quotient 8 *is the units' figure of the root,* or a figure greater than the units' figure. This quotient figure can never be too small, since the part 118 will be at *least equal to twice the product of the tens by the units:* but it may be too large; for, the 118 besides the double product of the tens by the units, may likewise contain tens arising from the square of the units.

To ascertain if the quotient 8 expresses the units, we write the 8 to the right of the 14, which gives 148, and then we multiply 148 by 8. Thus, we evidently form, 1st, the square of the units, and 2d, the double product of the tens by the units. This multiplication being effected, gives for a product 1184, a number equal

to the result of the first operation. Having subtracted the product, we find the remainder equal to 0: hence 78 is the root required.

Indeed, in the operations, we have merely subtracted from the iven number 6084, 1st, the square of 7 tens or of 70; 2d, twice the product of 70 by 8; and 3d, the square of 8: that is, the three parts which enter into the composition of the square of 78.

REMARK.—The operations in the last example have been performed on but two periods. It is plain, however, that the same reasoning is equally applicable to larger numbers; for, by changing the order of the units, we do not change the relation in which they stand to each other.

Thus, in the number 60 84 95, the two periods 60 84, have the same relation to each other, as in the number 6084; and hence, the methods pursued in the last example are equally applicable to larger numbers.

Hence, for the extraction of the square root of numbers, we have the following

RULE.

I. *Separate the given number into periods of two figures each, beginning at the right hand: the period on the left will often contain but one figure.*

II. *Find the greatest square in the first period on the left, and place its root on the right after the manner of a quotient in division. Subtract the square of the root from the first period, and to the remainder bring down the second period for a dividend.*

III. *Double the root already found and place it on the left for a divisor. Seek how many times the divisor is contained in the dividend, exclusive of the right-hand figure, and place the figure in the root and also at the right of the divisor.*

IV. *Multiply the divisor thus augmented, by the last figure of the root, and subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.*

V. *Double the whole root already found, for a new divisor, and continue the operation as before, until all the periods are brought down.*

I. REMARK.—If, after all the periods are brought down, there is no remainder, the proposed number is a perfect square. But if

there is a remainder, we have only found the root of the greatest perfect square contained in the given number, or *the entire part of the root sought.*

For example, if it were required to extract the square root of 168, we should find 12 for the entire part of the root and a remainder of 24, which shows that 168 is not a perfect square. But is the square of 12 the greatest perfect square contained in 168? That is, is 12 the entire part of the root? To prove this, we will first show that, *the difference between the squares of two consecutive numbers, is equal to twice the less number augmented by unity.*

$$\begin{array}{lrl}
\text{Let} & a & = \text{the less number,} \\
\text{and} & a+1 & = \text{the greater.} \\
\text{Then} & (a+1)^2 & = a^2+2a+1 \\
\text{and} & (a)^2 & = a^2 \\
\text{Their difference is} & & = \quad 2a+1 \quad \text{as enunciated.}
\end{array}$$

Hence, *the entire part of the root cannot be augmented by* 1, *unless the remainder is equal to, or exceeds twice the root found, plus unity.*

But, $12 \times 2 + 1 = 25$; and since the remainder 24 is less than 25, it follows that 12 cannot be augmented by a number as great as unity: hence, it is the entire part of the root.

The principle demonstrated above, may be readily applied in finding the squares of consecutive numbers.

If the numbers are large, it will be much easier to apply the above principle than to square the numbers separately.

For example, if we have $(651)^2 = 423801$;

and wish to find the square of 652, we have

$$\begin{array}{lrl}
 & (651)^2 & = 423801 \\
 & +2\times 651 & = \quad 1302 \\
 & +1 & = \qquad\ 1 \\
\text{and} & (652)^2 & = 425104. \\
\text{Also,} & (652)^2 & = 425104 \\
 & +2\times 652 & = \quad 1304 \\
 & +1 & = \qquad\ 1 \\
 & (653)^2 & = 426409.
\end{array}$$

II. REMARK.—The number of figures in the root will always be equal to the number of periods into which the given number is separated.

EXAMPLES.

1. To find the square root of 7225.
2. To find the square root of 17689.
3. To find the square root of 994009.
4. To find the square root of 85678973.
5. To find the square root of 67812675.

Of Incommensurable Numbers.

118. If a number is not a perfect square, its square root is said to be *incommensurable*, or *irrational*, because it cannot be expressed in terms of the numerical unit. Thus, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{7}$, are incommensurable numbers. They are also sometimes called *radicals* or *surds*.

Two or more numbers are said to be *prime* with respect to each other, when there is no whole number except unity which will divide each of them without a remainder. Thus, the numbers 3 and 5 are prime with respect to each other; and so also are 4 and 7 and 9.

In order to prove that the root of an imperfect power cannot be expressed by exact parts of unity, we must first show that,

Every number P, *which will exactly divide the product* A × B *of two numbers, and which is prime with one of them, will divide the other.*

Let us suppose that P will not divide A, and that A is greater than P.

Let us now find the greatest common divisor of A and P. If we represent the entire quotients by Q, Q′, Q″, &c., and the remainders, respectively, by R, R′, R″, &c.; we shall have

$$\mathrm{A}\ \left\|\frac{\mathrm{P}}{\mathrm{Q}}\right., \quad \text{hence,} \quad \mathrm{A} = \mathrm{PQ} + \mathrm{R},$$

$$\mathrm{P}\ \left\|\frac{\mathrm{R}}{\mathrm{Q'}}\right., \quad \text{hence,} \quad \mathrm{P} = \mathrm{RQ'} + \mathrm{R'},$$

$$\mathrm{R}\ \left\|\frac{\mathrm{R'}}{\mathrm{Q''}}\right., \quad \text{hence,} \quad \mathrm{R} = \mathrm{R'Q''} + \mathrm{R''},$$

$$\mathrm{R'}\ \left\|\frac{\mathrm{R''}}{\mathrm{Q'''}}\right., \quad \text{hence,} \quad \mathrm{R'} = \mathrm{R''Q'''} + \mathrm{R'''}.$$

Now, since he emainders R, R′, R″, &c., constantly diminish, if the division be continued sufficiently far, we shall obtain a remainder equal to unity; for the remainder cannot be 0, since by hypothesis A and P are prime with each other. Hence, we have the following equations:

$$\begin{aligned} A &= P\,Q + R \\ P &= R\,Q' + R' \\ R &= R'\,Q'' + R'' \\ R' &= R''Q''' + R''' \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

Multiplying the first of these equations by B, and dividing by P, we have

$$\frac{AB}{P} = BQ + \frac{BR}{P}.$$

But, by hypothesis, $\frac{AB}{P}$ is an entire number, and since B and Q are entire numbers, the product BQ is an entire number. Hence, it follows that $\frac{BR}{P}$ is an entire number.

If we multiply the second of the above equations by B, and divide by P, we have

$$B = \frac{BRQ'}{P} + \frac{BR'}{P}.$$

But we have already shown, that $\frac{BR}{P}$ is an entire number; hence $\frac{BRQ'}{P}$ is an entire number. This being the case, $\frac{BR'}{P}$ must also be an entire number. If the operation be continued until the number which multiplies B becomes 1, we shall have $\frac{B \times 1}{P}$ equal to an entire number, which proves that P will divide B.

In the operations above we have supposed $A > P$; but if $P > A$, we should first divide P by A.

Hence, *if a number* P *will exactly divide the product of two numbers, and is prime with one of them, it will divide the other.*

We see from what has preceded that, *if* P *is prime with respect to any number as* a, *it will also be prime with respect to* a^2 *and the higher powers of* a.

For, if P will divide $a^2 = a \times a$, it must divide one of the factors a or a. But this would be contrary to the supposition; hence, P cannot divide a^2. In the same way it may be proved that it cannot divide the higher powers of a.

We will now show that the square root of an imperfect square cannot be expressed by a fractional number.

Let c be an imperfect square. Then if its exact root can be expressed by a fractional number, we can assume

$$\sqrt{c} = \frac{a}{b},$$

in which the fraction $\frac{a}{b}$ is in its lowest terms: that is, in which a and b are prime with respect to each other.

Now, if we square both members of the equation, we have

$$c = \frac{a^2}{b^2},$$

in which c is an entire number; and hence, if the equation is true, a^2 must be divisible by b^2.

But if a^2 is divisible by b^2, the product $a \times a = a^2$, must be divisible by b; for the division would be effected by dividing twice by b. But we have seen that a^2 is not divisible by b; therefore, we cannot express the square root of an imperfect square by a fractional number.

Extraction of the Square Root of Fractions.

119. Since the second power of a fraction is obtained by squaring the numerator and denominator separately, it follows that the square root of a fraction will be equal to the square root of the numerator divided by the square root of the denominator.

For example, $$\sqrt{\frac{a^2}{b^2}} = \frac{a}{b},$$

since $$\frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}.$$

But if the numerator and the denominator are not both perfect squares, the root of the fraction cannot be exactly found. We can, however, easily find the exact root to within less than one of the equal parts of the fraction. For this purpose,

Multiply both terms of the fraction by the denominator—this makes the denominator a perfect square. Then extract the square root of the perfect square nearest the value of the numerator, and place the root of the denominator under it—this fraction will be the approximate root.

Thus, if it be required to extract the square root of $\frac{3}{5}$, we multiply both terms by 5, which gives $\frac{15}{25}$: the square nearest 15 is 16: hence, $\frac{4}{5}$ is the required root, and is exact to within less than $\frac{1}{5}$.

120. We may, by a similar method, determine, approximatively, the roots of whole numbers which are not perfect squares. Let it be required, for example, to determine the square root of an entire number a, nearer than the fraction $\frac{1}{n}$; that is to say, to find a number which shall differ from the exact root of a, by a quantity less than $\frac{1}{n}$. It may be observed that,

$$a = \frac{an^2}{n^2}.$$

If we designate by r the entire part of the root of an^2, the number an^2 will then be comprised between r^2 and $(r+1)^2$; and $\frac{an^2}{n^2}$ will be comprised between $\frac{r^2}{n^2}$ and $\frac{(r+1)^2}{n^2}$; and consequently the true root of a is comprised between

$$\sqrt{\frac{r^2}{n^2}} \text{ and } \sqrt{\frac{(r+1)^2}{n^2}}.$$

that is, between $\frac{r}{n}$ and $\frac{r+1}{n}$. But the difference between these numbers is $\frac{1}{n}$: hence $\frac{r}{n}$ will represent the square root

of a within less than the fraction $\frac{1}{n}$. Hence to obtain the root:

Multiply the given number by the square of the denominator of the fraction which determines the degree of approximation: then extract the square root of the product to the nearest unit, and divide this root by the denominator of the fraction.

1. Suppose, for example, it were required to extract the square root of 59, to within less than $\frac{1}{12}$.

First, $(12)^2 = 144$; and $144 \times 59 = 8496$.

Now, the square root of 8496 to the nearest unit, is 92: hence $\frac{92}{12} = 7\tfrac{8}{12}$, which is true to within less than $\frac{1}{12}$.

2. To find the $\sqrt{11}$ to within less than $\frac{1}{15}$.

Ans. $3\frac{4}{15}$.

3 To find the $\sqrt{223}$ to within less than $\frac{1}{40}$.

Ans. $14\frac{37}{40}$.

121. The manner of determining the approximate root in decimals, is a consequence of the preceding rule.

To obtain the square root of an entire number within $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c., it is only necessary according to the preceding rule, to multiply the proposed number by $(10)^2$, $(100)^2$, $(1000)^2$; or, which is the same thing,

Annex to the number, two, four, six, &c., ciphers: then extract the root of the product to the nearest unit, and divide this root by 10, 100, 1000, &c., *which is effected by pointing off one, two, three, &c., decimal places from the right hand.*

EXAMPLES.

1. To find the square root of 7 to within $\frac{1}{100}$.

Having multiplied by $(100)^2$, that is, having annexed four ciphers to the right hand of 7, it becomes 70000, whose root extracted to the nearest unit, is 264, which being divided by 100 gives 2.64 for the answer, which is true to within less than $\frac{1}{100}$.

```
      7ȯȯ0ȯ0 | 2.64
      4      |
 46 | 300
    | 276
524 |  2400
    |  2096
        304 Rem.
```

2. Find the $\sqrt{29}$ to within $\frac{1}{100}$. *Ans.* 5.38.

3. Find the $\sqrt{227}$ to within $\frac{1}{10000}$. *Ans.* 15.0665.

Remark.—The number of ciphers to be annexed to the whole number, is always double the number of decimal places required to be found in the root.

122. The manner of extracting the square root of decimal fractions is deduced immediately from the preceding article.

Let us take for example the number 3.425. This fraction is equivalent to $\frac{3425}{1000}$. Now 1000 is not a perfect square, but the denominator may be made such without altering the value of the fraction, by multiplying both the terms by 10; this gives $\frac{34250}{10000}$ or $\frac{34250}{(100)^2}$. Then extracting the square root of 34250 to the nearest unit, we find 185; hence $\frac{185}{100}$ or 1.85 is the required root to within $\frac{1}{100}$.

If greater exactness be required, it will be necessary to add to the number 3.4250 so many ciphers as shall make the periods of decimals equal to the number of decimal places to be found in the root. Hence, to extract the square root of a decimal fraction:

Annex ciphers to the proposed number until the number of decimal places shall be equal to double the number required in the root. Then extract the root to the nearest unit, and point off from the right hand the required number of decimal places.

EXAMPLES.

1. Find the $\sqrt{3271.4707}$ to within .01. *Ans.* 57.19.

2. Find the $\sqrt{31.027}$ to within .01. *Ans.* 5.57.

3. Find the $\sqrt{0.01001}$ to within .00001. *Ans.* 0.10004.

123. Finally, if it be required to find the square root of a vulgar fraction in terms of decimals:

Change the vulgar fraction into a decimal and continue the division until the number of decimal places is double the number required in the root. Then extract the root of the decimal by the last rule.

EXAMPLES.

1. Extract the square root of $\frac{11}{14}$ to within .001. This number, reduced to decimals, is 0.785714 to within 0.000001. The root of 0.785714 to the nearest unit, is .886: hence 0.886 is the root of $\frac{11}{14}$ to within .001.

2. Find the $\sqrt{2\frac{13}{15}}$ to within 0.0001. *Ans.* 1.6931.

Extraction of the Square Root of Algebraic Quantities.

124. Let us first consider the case of a monomial. In order to discover the process for extracting the square root, let us see how the square of a monomial is formed.

By the rule for the multiplication of monomials (Art. 41), we have

$$(5a^2b^3c)^2 = 5a^2b^3c \times 5a^2b^3c = 25a^4b^6c^2;$$

that is, in order to square a monomial, it is necessary to *square its co-efficient, and double the exponent of each letter.* Hence, to find the square root of a monomial,

1st. *Extract the square root of the co-efficient and divide the exponent of each letter by two.* 2d. *To the root of the co-efficient annex each letter with its new exponent, and the result will be the required root.*

Thus, $\sqrt{64a^6b^4} = 8a^3b^2$; for, $8a^3b^2 \times 8a^3b^2 = 64a^6b^4$,

and, $\sqrt{625a^2b^8c^6} = 25ab^4c^3$; for, $(25ab^4c^3)^2 = 625a^2b^8c^6$.

125. From the preceding rule, it follows, that, when a monomial is a perfect square, *its numerical co-efficient is a perfect square, and the exponent of every letter an even number.* Thus, $25a^4b^2$ is a perfect square, but $98ab^4$ is not a perfect square; for, 98 is not a perfect square, and a is affected with an uneven exponent.

An imperfect square is introduced into the calculus by affecting it with the radical sign $\sqrt{}$, and written thus, $\sqrt{98ab^4}$. Quantities of this kind are called *radical quantities*, or *irrational quantities*, or simply *radicals of the second degree.*

These expressions may sometimes be simplified. For, by the definition of the square root, we have

$$\begin{aligned}
\sqrt{a} \times \sqrt{a} &= (\sqrt{a})^2 = a,\\
\sqrt{ab} \times \sqrt{ab} &= (\sqrt{ab})^2 = ab,\\
\sqrt{abc} \times \sqrt{abc} &= (\sqrt{abc})^2 = abc,\\
\sqrt{abcd} \times \sqrt{abcd} &= (\sqrt{abcd})^2 = abcd;
\end{aligned}$$

and the same would be true for any number of factors.

Again,

$$(\sqrt{a}\,.\sqrt{b}\,.\sqrt{c}\,.\sqrt{d})^2 = (\sqrt{a})^2.(\sqrt{b})^2.(\sqrt{c})^2.(\sqrt{d})^2 = abcd,$$

by the rule for multiplying monomials (Art. 41).

Now, since, $(\sqrt{abcd})^2 = abcd,$

and, $(\sqrt{a}\,.\sqrt{b}\,.\sqrt{c}\,.\sqrt{d})^2 = abcd;$

it follows, that the quantities themselves are equal: hence,

$$\sqrt{abcd} = \sqrt{a}\,.\sqrt{b}\,.\sqrt{c}\,.\sqrt{d}; \text{ that is,}$$

The square root of the product of two or more factors is equal to the product of the square roots of those factors.

This being proved, we can write

$$\sqrt{98ab^4} = \sqrt{49b^4 \times 2a} = \sqrt{49b^4} \times \sqrt{2a}.$$

But, $\sqrt{49b^4} = 7b^2$:

hence, $\sqrt{98ab^4} = 7b^2\sqrt{2a}.$

In like manner,

$$\sqrt{45a^2b^3c^2d} = \sqrt{9a^2b^2c^2 \times 5bd} = 3abc\sqrt{5bd}.$$

$$\sqrt{864a^2b^5c^{11}} = \sqrt{144a^2b^4c^{10} \times 6bc} = 12ab^2c^5\sqrt{6bc}.$$

The quantity which stands without the radical sign is called the *co-efficient* of the radical. Thus, $7b^2$, $3abc$, and $12ab^2c^5$, are *co-efficients of the radicals.*

In general, to simplify a radical of the second degree:

Divide the quantity under the radical sign by the smallest monomial, with reference to its co-efficients and exponents, that will give for a quotient a perfect square. Then, extract the root of the perfect square and place it without the radical sign, under which, write the monomial used as a divisor.

EXAMPLES.

1. To reduce $\sqrt{75a^3bc}$ to its simplest form. $= 5a\sqrt{3abc}$
2. To reduce $\sqrt{128b^5a^6d^2}$ to its simplest form. $= 8a^3b^2d\sqrt{2b}$
3. To reduce $\sqrt{32a^9b^8c}$ to its simplest form. $= 4a^4b^4\sqrt{2ac}$
4. To reduce $\sqrt{256a^2b^4c^8}$ to its simplest form. $= 16ab^2c^4$
5. To reduce $\sqrt{1024a^9b^7c^5}$ to its simplest form. $= 32a^4b^3c^2\sqrt{abc}$
6. To reduce $\sqrt{728a^7b^5c^6d}$ to its simplest form. =

126. Since like signs in both the factors give a plus sign in the product, the square of $-a$, as well as that of $+a$, will be a^2: hence, the root of a^2 is either $+a$ or $-a$. Also, the square root of $25a^2b^4$ is either $+5ab^2$ or $-5ab^2$. Whence we may conclude, that if a monomial is positive, its square root may be affected either with the sign $+$ or $-$;

thus, $\sqrt{9a^4} = \pm 3a^2$,

for, $+3a^2$ or $-3a^2$, squared, gives $9a^4$. The double sign $\pm$ with which the root is affected, is read *plus* or *minus*.

If the proposed monomial were *negative*, it would have no square root, since it has just been shown that the square of every quantity, whether positive or negative, is essentially positive. Therefore,

$$\sqrt{-9}, \quad \sqrt{-4a^2}, \quad \sqrt{-8a^2b},$$

are algebraic symbols which indicate operations that cannot be

performed. They are called *imaginary quantities*, or rather, *imaginary expressions*, and are frequently met with in the resolution of equations of the second degree.

127. Let us now examine the *law of formation* of the square of a polynomial; for, from this law, the rule is deduced for extracting the square root.

It has already been shown (Art. 46), that,

$$(a + b)^2 = a^2 + 2ab + b^2; \text{ that is,}$$

The square of a binomial is equal to the square of the first term plus twice the product of the first term by the second, plus the square of the second.

The square of a polynomial, is the product arising from multiplying the polynomial by itself once: hence, the *first term* of the product, arranged with reference to a particular letter, is the square of the first term of the polynomial arranged with reference to the same letter. Therefore, the square root of the first term of such a product will be the first term of the required root.

128. Let us now extract the square root of the polynomial

$$28a^5 + 49a^4 + 4a^6 + 9 + 42a^2 + 12a^3,$$

which arranged with reference to the letter a, becomes,

$4a^6 + 28a^5 + 49a^4 + 12a^3 + 42a^2 + 9$	$2a^3 + 7a^2 + 3$
$4a^6$	$4a^3 + 7a^2$
$R = 28a^5 + 49a^4 + 12a^3 + 42a^2 + 9$	$7a^2$
$28a^5 + 49a^4$	$28a^5 + 49a^4 = (2r + r')r'$
$R' = \quad - \quad - \quad 12a^3 + 42a^2 + 9$	$4a^3 + 14a^2 + 3$
$12a^3 + 42a^2 + 9$	3
$R'' = \quad - \quad - \quad 0 \quad 0 \quad 0$	$12a^2 + 42a^2 + 9 = (2n + r'')r''$

Now, since the square root of $4a^6$ is $2a^3$, it follows that $2a^3$ is the first term of the required root. Designate this term by r, and the following terms of the root, arranged with reference to a, by r', r'', r''', &c.

Now, if we denote the given polynomial by N, we shall have

$$N = (r + r' + r'' + r''' + \&c.;)^2$$

or, if we designate all the terms of the root, after the first, by s

$$N = (r + s)^2 = r^2 + 2rs + s^2$$
$$= r^2 + 2r(r' + r'' + r''' + \&c.) + s^2.$$

If now we subtract $r^2 = 4a^6$, from N, and designate the remainder by R, we shall have

$$R = N - 4a^6 = 2r\,(r' + r'' + r''' + \&c.\,) + s^2;$$

in which the first term $2rr'$ will contain a to a higher power than either of the following terms. Hence, *if the first term of the first remainder be divided by twice the first term of the root, the quotient will be the second term of the root.*

If now, we place $\quad r + r' = n$

and designate the remaining terms of the root, r'', r''', &c., by s', we shall have

$$N = (n + s')^2 = n^2 + 2ns' + s'^2; \text{ and}$$

$$R' = N - n^2 = (2r + 2r')\,(r'' + r''' + \&c.) + s''^2;$$

in which, if we perform the multiplications indicated in the second member, the term $2rr''$ will contain a higher power of a than either of the following terms. *Hence, if the first term of the second remainder be divided by twice the first term of the root, the quotient will be the third term of the root.*

If we make

$$r + r' + r'' = n', \text{ and } r''' + r^{\text{IV}} + \&c. = s'',$$

we shall have

$$N = (n' + s'')^2 = n'^2 + 2n's'' + s''^2; \text{ and}$$

$$R'' = N - n'^2 = 2\,(r + r' + r'')\,(r''' + r^{\text{IV}} + \&c.) + s''^2;$$

from which we see, that *the first term of any remainder, divided by twice the first term of the root, will give a new term of the required root.*

It should be observed, that instead of subtracting n^2 from the given polynomial, in order to find the second remainder, that that remainder could be found by subtracting $(2r + r')\,r'$ from the first remainder. So the third remainder may be found by subtracting $(2n + r'')\,r''$ from the second, and similarly for the remainders which follow.

In the example above, the third remainder is equal to zero, and hence the given polynomial has an exact root.

Hence, for the extraction of the square root of a polynomial, we have the following

$n = r + r'$

$R' = N - n^2 = N - (r + r')^2 = $ [illegible] $2rr' + r'^2$

$= N - r^2 - (2r + r')\,r' = R - (2r + r')\,r'$

RULE.

I. *Arrange the polynomial with reference to one of its letters, and then extract the square root of the first term, which will give the first term of the root. Subtract the square of this term from the given polynomial.*

II. *Divide the first term of the remainder by twice the first term of the root, and the quotient will be the second term of the root.*

III. *From the first remainder subtract the product of twice the first term of the root plus the second term, by the second term.*

IV. *Divide the first term of the second remainder by twice the first term of the root, and the quotient will be the third term of the root.*

V. *From the second remainder subtract the product of twice the first and second terms of the root, plus the third term by the third term, and the result will be the third remainder, from which the fourth term of the root may be found; and proceed in a similar manner for the remaining terms of the root.*

EXAMPLES.

1. Extract the square root of the polynomial

$$49a^2b^2 - 24ab^3 + 25a^4 - 30a^3b + 16b^4.$$

First arrange it with reference to the letter a.

$25a^4 - 30a^3b + 49a^2b^2 - 24ab^3 + 16b^4$	$5a^2 - 3ab + 4b^2$
$25a^4$	$10a^2 - 3ab$
$R = -30a^3b + 49a^2b^2 - 24ab^3 + 16b^4$	$-3ab$
$-30a^3b + 9a^2b^2$	$30a^3b + 9a^2b^2$
$R' = +40a^2b^2 - 24ab^3 + 16b^4$	$10a^2 - 6ab + 4b^2$
$+40a^2b^2 - 24ab^3 + 16b^4$	$4b^2$
$R'' =$ - - - - - -	$40a^2b^2 - 24ab^3 + 16b^4$

2. Find the square root of

$$a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4.$$

3. Find the square root of

$$a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4.$$

4. Find the square root of

$$4x^6 + 12x^5 + 5x^4 - 2x^3 + 7x^2 - 2x + 1.$$

5. Find the square root of

$$9a^4 - 12a^3b + 28a^2b^2 - 16ab^3 + 16b^4.$$

6. Find the square root of

$$25a^4b^2 - 40a^3b^2c + 76a^2b^2c^2 - 48ab^2c^3 + 36b^2c^4 - 30a^4bc + 24a^3bc^2 - 36a^2bc^3 + 9a^4c^2.$$

129. We will conclude this subject with the following remarks:

1st. A binomial can never be a perfect square. For, its root cannot be a monomial, since the square of a monomial will be a monomial; nor can the root be a polynomial, since the square of the simplest polynomial, viz., a binomial, will contain at least three terms. Thus, an expression of the form

$$a^2 \pm b^2$$

can never be a perfect square.

2d. A trinomial, however, may be a perfect square. If so, when arranged, its two extreme terms must be squares, and the middle term double the product of the square roots of the other two. Therefore, to obtain the square root of a trinomial, when it is a perfect square,

Extract the roots of the two extreme terms, and give these roots the same or contrary signs, according as the middle term is positive or negative. To verify it, see if the double product of the two roots is equal to the middle term of the trinomial. Thus,

$$9a^6 - 48a^4b^2 + 64a^2b^4 \quad \text{is a perfect square,}$$

for, $\sqrt{9a^6} = 3a^3$, and $\sqrt{64a^2b^4} = -8ab^2$,

and also, $2 \times 3a^3 \times -8ab^2 = -48a^4b^2$, the middle term.

But $$4a^2 + 14ab + 9b^2$$

is not a perfect square: for, although $4a^2$ and $+9b^2$ are the squares of $2a$ and $3b$, yet $2 \times 2a \times 3b$ is not equal to $14ab$.

3d. When, in extracting the square root of a polynomial, the first term of any one of the remainders is not exactly divisible by twice the first term of the root, we may conclude that the proposed polynomial is not a perfect square. This is an evident consequence of the course of reasoning, from which the general rule for extracting the square root was deduced.

4th. When the polynomial is *not a perfect square*, the expression for its square root may sometimes be simplified.

Take, for example, the expression

$$\sqrt{a^3b + 4a^2b^2 + 4ab^3}.$$

The quantity under the radical is not a perfect square; but it can be put under the form

$$ab\,(a^2 + 4ab + 4b^2).$$

Now, the factor within the parenthesis is evidently the square of $a + 2b$, whence we have

$$\sqrt{a^3b + 4a^2b^2 + 4ab^3} = (a + 2b)\sqrt{ab}.$$

Of the Calculus of Radicals of the Second Degree.

130. A *radical quantity* is the indicated root of an imperfect power. If the root indicated is the square root, the expression is called *a radical of the second degree*. Thus,

$$\sqrt{a},\quad 3\sqrt{b},\quad 7\sqrt{2},$$

are radicals of the second degree.

131. Two radicals of the second degree are *similar*, when the quantities under the radical sign are the same in both. Thus, $3\sqrt{b}$ and $5c\sqrt{b}$ are similar radicals; and so also, are $9\sqrt{2}$ and $7\sqrt{2}$.

Addition and Subtraction.

132. In order to add or subtract similar radicals, *add or subtract their co-efficients, and to the sum or difference annex the common radical.*

Thus, $\quad 3a\sqrt{b} + 5c\sqrt{b} = (3a + 5c)\sqrt{b}$;

and $\quad 3a\sqrt{b} - 5c\sqrt{b} = (3a - 5c)\sqrt{b}$.

In like manner,

$$7\sqrt{2a} + 3\sqrt{2a} = (7 + 3)\sqrt{2a} = 10\sqrt{2a};$$

and $\quad 7\sqrt{2a} - 3\sqrt{2a} = (7 - 3)\sqrt{2a} = 4\sqrt{2a}.$

Two radicals, which do not appear to be similar, may become so by simplification (Art. 125).

For example,

$$\sqrt{48ab^2} + b\sqrt{75a} = 4b\sqrt{3a} + 5b\sqrt{3a} = 9b\sqrt{3a};$$

and $2\sqrt{45} - 3\sqrt{5} = 6\sqrt{5} - 3\sqrt{5} = 3\sqrt{5}$.

When the radicals are not similar, their addition or subtraction can only be indicated. Thus, to add $3\sqrt{b}$ to $5\sqrt{a}$, we write

$$5\sqrt{a} + 3\sqrt{b}.$$

Multiplication.

133. To multiply one radical by another, let us observe that

$$(\sqrt{a} \times \sqrt{b})^2 = ab; \text{ also}$$

that, $$(\sqrt{ab})^2 = ab; \text{ hence,}$$

$$(\sqrt{a} \times \sqrt{b})^2 = (\sqrt{ab})^2; \text{ and consequently,}$$

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}: \text{ that is}$$

The product of two radicals of the second degree is equal to the square root of the product of the quantities under the radical signs.

When there are co-efficients, *we first multiply them together, and write the product before the radical sign.* Thus

$$3\sqrt{5ab} \times 4\sqrt{20a} = 12\sqrt{100a^2b} = 120a\sqrt{b}$$

$$2a\sqrt{bc} \times 3a\sqrt{bc} = 6a^2\sqrt{b^2c^2} = 6a^2bc.$$

$$2a\sqrt{a^2+b^2} \times -3a\sqrt{a^2+b^2} = -6a^2(a^2+b^2).$$

Division.

134. To divide one radical by another, let us observe that

$$\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{(\sqrt{a})^2}{(\sqrt{b})^2} = \frac{a}{b}: \text{ also,}$$

that $$\left(\sqrt{\frac{a}{b}}\right)^2 = \frac{a}{b}: \text{ hence,}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}: \text{ that is,}$$

The quotient of two radicals is equal to the square root of the quotient of the quantities under the radical signs.

When there are co-efficients, *write their quotient as the co-efficient of the radicals.*

For example,

$$5a\sqrt{b} \div 2b\sqrt{c} = \frac{5a}{2b}\sqrt{\frac{b}{c}}.$$

And $$12ac\sqrt{6bc} \div 4c\sqrt{2b} = 3a\sqrt{\frac{6bc}{2b}} = 3a\sqrt{3c}.$$

135. There are *two transformations* of frequent use in finding the numerical values of radicals.

The first transformation consists in passing the co-efficient of a radical under the radical sign. Take, for example, the expression $3a\sqrt{5b}$. By applying the rules for the multiplication of radicals we may write,

$$3a\sqrt{5b} = \sqrt{9a^2} \times \sqrt{5b} = \sqrt{9a^2 \times 5b} = \sqrt{45a^2b}.$$

Therefore, *the co-efficient of a radical may be passed under the radical sign, as a factor, by squaring it.*

The principal use of this transformation, is to find an approximate value of any radical, which shall differ from its true value, by less than unity.

For example, take the expression $6\sqrt{13}$. Now, as 13 is not a perfect square, we can only find an approximate value for its square root; and when this approximate value is multiplied by 6, the product will differ materially from the true value of $6\sqrt{13}$. But if we write,

$$6\sqrt{13} = \sqrt{6^2 \times 13} = \sqrt{36 \times 13} = \sqrt{468},$$

we find that the square root of 468 is the whole number 21, plus an irrational number less than unity. Hence,

$6\sqrt{13} = 21$ plus an irrational number less than 1.

In a similar manner we may find,

$12\sqrt{7} = 31$ plus an irrational number less than 1.

136. Having given an expression of the form,

$$\frac{a}{p + \sqrt{q}} \text{ or } \frac{a}{p - \sqrt{q}},$$

in which a and p are any numbers whatever, and q not a perfect square, it is the object of the second transformation to render the denominator a rational quantity.

This object is attained by multiplying both terms of the fraction by $p - \sqrt{q}$, when the denominator is $p + \sqrt{q}$, and by $p + \sqrt{q}$, when the denominator is $p - \sqrt{q}$; and recollecting that the sum of two quantities, multiplied by their difference, is equal to the difference of their squares: hence,

$$\frac{a}{p+\sqrt{q}} = \frac{a(p-\sqrt{q})}{(p+\sqrt{q})(p-\sqrt{q})} = \frac{a(p-\sqrt{q})}{p^2-q} = \frac{ap-a\sqrt{q}}{p^2-q},$$

$$\frac{a}{p-\sqrt{q}} = \frac{a(p+\sqrt{q})}{(p-\sqrt{q})(p+\sqrt{q})} = \frac{a(p+\sqrt{q})}{p^2-q} = \frac{ap+a\sqrt{q}}{p^2-q},$$

in which the denominators are rational.

As an example to illustrate the utility of this method of approximation, let it be required to find the approximate value of the expression $\frac{7}{3-\sqrt{5}}$. We write

$$\frac{7}{3-\sqrt{5}} = \frac{7(3+\sqrt{5})}{9-5} = \frac{21+7\sqrt{5}}{4}.$$

But, $\quad 7\sqrt{5} = \sqrt{49 \times 5} = \sqrt{245} = 15 +$ an irrational number less than unity. Therefore,

$$\frac{7}{3-\sqrt{5}} = \frac{21 + 15 + \text{ irr. number } < 1}{4} = 9 + < \frac{1}{4};$$

hence, 9 differs from the true value by less than *one fourth.*

If we wish a more exact value for this expression, *extract the square root of* 245 *to a certain number of decimal places, add* 21 *to this root, and divide the result by* 4.

For another example, take

$$\frac{7\sqrt{5}}{\sqrt{11}+\sqrt{3}},$$

and find its value to within 0.01.

We have,

$$\frac{7\sqrt{5}}{\sqrt{11}+\sqrt{3}} = \frac{7\sqrt{5}(\sqrt{11}-\sqrt{3})}{11-3} = \frac{7\sqrt{55}-7\sqrt{15}}{8},$$

Now, $7\sqrt{55} = \sqrt{55 \times 49} = \sqrt{2695} = 51.91$, within 0.01,

and $7\sqrt{15} = \sqrt{15 \times 49} = \sqrt{735} = 27.11$ - - - ;

therefore,

$$\frac{7\sqrt{5}}{\sqrt{11} + \sqrt{3}} = \frac{51.91 - 27.11}{8} = \frac{24.80}{8} = 3.10.$$

Hence, we have 3.10 for the required result. This is exact to within $\frac{1}{800}$.

By a similar process, it will be found that

$$\frac{3 + 2\sqrt{7}}{5\sqrt{12} - 6\sqrt{5}} = 2{,}123, \text{ exact to within } 0.001.$$

REMARK.—The value of expressions similar to those above, may be calculated by approximating to the value of each of the radicals which enter the numerator and denominator. But as the value of the denominator would not be exact, we could not determine the degree of approximation which would be obtained, whereas by the method just indicated, the denominator becomes *rational*, and we always know to what degree the approximation is made.

Examples in the Calculus of Radicals.

We make the reductions in the examples which follow, according to the methods indicated in Art. 125; though, it is sometimes necessary to multiply the quantity under the radical sign, instead of dividing it.

1. Reduce $\sqrt{125}$ to its simplest form.

We first seek the largest perfect square that will exactly divide 125. We try, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, and 144. We find that 25 is the only one that will give an exact quotient: hence,

$$\sqrt{125} = \sqrt{25 \times 5} = 5\sqrt{5}.$$

2. Reduce $\sqrt{\frac{50}{147}}$ to its simplest form.

We observe that 25 will divide the numerator, and hence

$$\sqrt{\frac{50}{147}} = \sqrt{\frac{25 \times 2}{147}} = 5\sqrt{\frac{2}{147}}.$$

Since there is no perfect square which will divide 147, we must see if we can multiply it by any number which will give a perfect square for a product. Multiplying by 2 we have 284, which is not a perfect square. Then trying 3, we find the product 441, whose square root is 21. Hence, we have

$$5\sqrt{\frac{2}{147}} = 5\sqrt{\frac{2 \times 3}{147 \times 3}} = 5\sqrt{\frac{1}{441} \times 6} = \frac{5}{21}\sqrt{6}.$$

3. Reduce $\sqrt{98a^2x}$ to its most simple form.

Ans. $7a\sqrt{2x}$.

4. Reduce $\sqrt{(x^3 - a^2x^2)}$ to its most simple form.

5. Required the sum of $\sqrt{72}$ and $\sqrt{128}$.

Ans. $14\sqrt{2}$.

6. Required the sum of $\sqrt{27}$ and $\sqrt{147}$.

Ans. $10\sqrt{3}$.

7. Required the sum of $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{27}{50}}$.

Ans. $\frac{19}{30}\sqrt{6}$.

8. Required the sum of $2\sqrt{a^2b}$ and $3\sqrt{64bx^4}$.

9. Required the sum of $9\sqrt{243}$ and $10\sqrt{363}$.

10. Required the difference of $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{5}{27}}$.

Ans. $\frac{14}{45}\sqrt{15}$.

11. Required the product of $5\sqrt{8}$ and $3\sqrt{5}$.

Ans. $30\sqrt{10}$.

12. Required the product of $\frac{2}{3}\sqrt{\frac{1}{8}}$ and $\frac{3}{4}\sqrt{\frac{7}{10}}$.

Ans. $\frac{1}{40}\sqrt{35}$.

13. Divide $6\sqrt{10}$ by $3\sqrt{5}$.

14. What is the sum of $\sqrt{48ab^2} + b\sqrt{75a}$.

15. What is the sum of $\sqrt{18a^5b^3} + \sqrt{50a^3b^3}$.

Ans. $(3a^2b + 5ab)\sqrt{2ab}$.

CHAPTER VI.

EQUATIONS OF THE SECOND DEGREE.

137. An Equation of the second degree is one in which the greatest exponent of the unknown quantity is equal to 2. If the equation contains two unknown quantities, it is of the second degree when the greatest sum of the exponents with which the unknown quantities are affected, in any term, is equal to 2.

138. Equations of the second degree are divided into two classes.

1st. Equations which involve only the square of the unknown quantity and known terms. These are called *incomplete equations.*

2d. Equations which involve the first and second powers of the unknown quantity and known terms. These are called *complete equations.*

Thus, $$x^2 + 2x^2 - 5 = 7$$

and $$5x^2 - 3x^2 - 4 = a,$$

are incomplete equations; and

$$3x^2 - 5x - 3x^2 + a = b$$

$$2x^2 - 8x^2 - x - c = d,$$

are complete equations.

Of Incomplete Equations.

139. The following is the most general form of an incomplete equation: viz.,

$$ax^2 - \frac{bx^2}{c} - \frac{7}{6} = d.$$

If we reduce this to an equation containing only entire terms, we have,

$$6acx^2 - 6bx^2 - 7c = 6cd \cdot$$

hence, $x^2(6ac - 6b) = 6cd + 7c$,

and, $x^2 = \dfrac{6cd + 7c}{6ac - 6b} = m$,

by substituting m, for the known terms which compose the second member. Hence, *every incomplete equation can be reduced to an equation involving but two terms,* of the form

$$x^2 = m;$$

and from this circumstance, the incomplete equations are often called, *equations involving two terms.*

There is no difficulty in resolving equations of this form; for,

we have $x = \sqrt{m}$.

If m is a perfect square, the exact value of x can be found by extracting its square root, and the value will be expressed either algebraically or in numbers.

If m is an algebraic quantity, and not a perfect square, it must be reduced to its simplest form by the rules for reducing radicals of the second degree. If m is a number, and not a perfect square, its square root must be determined approximatively by the rules already given.

But the *square* of any number is $+$, whether the number itself have the $+$ or $-$ sign: hence, it follows that

$$(+\sqrt{m})^2 = m, \text{ and } (-\sqrt{m})^2 = m;$$

and therefore, the unknown quantity x is susceptible of two distinct values, viz.

$$x = +\sqrt{m}, \text{ and } x = -\sqrt{m};$$

and either of these values being substituted for x will verify the given equation. For,

$$x \times x = x^2 = +\sqrt{m} \times +\sqrt{m} = m;$$

and $x \times x = x^2 = -\sqrt{m} \times -\sqrt{m} = m$.

The *root* of an equation is any expression which, being substituted for the unknown quantity, will satisfy the equation; that is, make the two members equal to each other. Hence, *every incomplete equation of the second degree has two roots which are numerically equal to each other; one having the sign plus, and the other the sign minus.*

1. Let us take, as an example, the equation

$$\frac{1}{3}x^2 - 3 + \frac{5}{12}x^2 = \frac{7}{24} - x^2 + \frac{299}{24}.$$

which, by making the terms entire, becomes

$$8x^2 - 72 + 10x^2 = 7 - 24x^2 + 299,$$

and by transposing and reducing

$$42x^2 = 378 \quad \text{and} \quad x^2 = \frac{378}{42} = 9;$$

hence, $x = +\sqrt{9} = +3$; and $x = -\sqrt{9} = -3$.

2. As a second example, let us take the equation

$$3x^2 = 5.$$

Dividing by 3 and extracting the square root, we have

$$x = \pm\sqrt{\frac{5}{3}} = \pm\frac{1}{3}\sqrt{15};$$

in which the values must be determined approximatively.

3. What are the values of x in the equation

$$11(x^2 - 4) = 5(x^2 + 2). \qquad \textit{Ans. } x = \pm 3.$$

4. What are the values of x in the equation

$$\frac{\sqrt{m^2 - x^2}}{x} = n \qquad \textit{Ans. } x = \pm\frac{m}{\sqrt{1 + n^2}}.$$

Of Complete Equations of the Second Degree.

140. The most general complete equation of the second degree can be expressed under the form

$$ax^2 + bx - 3x - c = d;$$

which may be put under the form

$$ax^2 + (b - 3)x = d + c;$$

and by dividing by the co-efficient of x^2, we have

$$x^2 + \frac{b - 3}{a}x = \frac{d + c}{a}.$$

If now, we substitute $2p$ for the co-efficient of x, and represent the value of the second member by q, we shall have,

$$x^2 + 2px = q.$$

The reduction to the above form is made:

1st. By transposing all the terms involving x^2 and x to the first member, and all the known terms to the second.

2d. If the term involving x^2 should be negative, the signs of all the terms of the equation must be changed to render it positive, and then divide both members by the co-efficient of x^2. Hence, *every complete equation of the second degree can be reduced to an equation involving but three terms, and of the above form.* The quantity q is called the *absolute term.*

If we compare the first member of the equation

$$x^2 + 2px = q,$$

with the square of a binomial

$$(x + a)^2 = x^2 + 2ax + a^2$$

we see that it needs but the square of p to render it a perfect square. If then, p^2 be added to the first member, it will become a perfect square; but in order to preserve the equality of the members, p^2 must also be added to the second member. Making these additions, we have

$$x^2 + 2px + p^2 = q + p^2;$$

this is called *completing the square,* which is done by adding the square of half the co-efficient of x to both members of the equation.

Now, if we extract the square root of both members, we have,

$$x + p = \pm\sqrt{q + p^2},$$

and by transposing p, we shall have

$$x = -p + \sqrt{q + p^2}, \quad \text{and} \quad x = -p - \sqrt{q + p^2}.$$

Either of these values being substituted for x in the equation

$$x^2 + 2px = q$$

will satisfy it. For, we have from the first value,

$$x^2 = (-p + \sqrt{q + p^2})^2 = p^2 - 2p\sqrt{q + p^2} + q + p^2$$

and

$$2px = 2p \times (-p + \sqrt{q + p^2}) = -2p^2 + 2p\sqrt{q + p^2}.$$

hence, $\qquad x^2 + 2px = q.$

For the second value, we have

$$x^2 = (-p - \sqrt{q + p^2})^2 = \qquad p^2 + 2p\sqrt{q + p^2} + q + p^2$$

and

$$2px = 2p\,(-p - \sqrt{q + p^2}) \; = -2p^2 - 2p\sqrt{q + p^2}\,;$$

hence, $\qquad x^2 + 2px = q\,;$

and consequently, the values found above, are *roots* of the equation.

In order to refer readily, to either of these values, we shall call the one which arises from using the $+$ sign before the radical, the *first value* of x, or the *first root* of the equation; and the other, the *second value* of x, or the *second root* of the equation.

Having reduced a complete equation of the second degree to the form

$$x^2 + 2px = q,$$

we can write immediately the two values of the unknown quantity by the following

RULE.

The first value of the unknown quantity is equal to half the co-efficient of x taken with a contrary sign, plus the square root of the absolute term increased by the square of half the co-efficient of x.

The second value of the unknown quantity is equal to half the co-efficient of x taken with a contrary sign, minus the square root of the absolute term increased by the square of half the co-efficient of x.

1. Let us take as an example the equation

$$x^2 - 7x + 10 = 0.$$

By transposing 10, we have

$$x^2 - 7x = -10.$$

Hence, $\qquad x = 3.5 + \sqrt{-10 + (3.5)^2} = 3.5 + \sqrt{2.25} = 5,$

and $\qquad x = 3.5 - \sqrt{-10 + (3.5)^2} = 3.5 - \sqrt{2.25} = 2.$

2. As a second example, let us take the equation

$$\frac{5}{6}x^2 - \frac{1}{2}x + \frac{3}{4} = 8 - \frac{2}{3}x - x^2 + \frac{273}{12}.$$

Reducing to entire terms, we have

$$10x^2 - 6x + 9 = 96 - 8x - 12x^2 + 273,$$

and transposing and reducing,

$$22x^2 + 2x = 360,$$

and dividing both members by 22,

$$x^2 + \frac{2}{22}x = \frac{360}{22};$$

hence, $$x = -\frac{1}{22} + \sqrt{\frac{360}{22} + \left(\frac{1}{22}\right)^2}$$

and $$x = -\frac{1}{22} - \sqrt{\frac{360}{22} + \left(\frac{1}{22}\right)^2}.$$

It often occurs, in the solution of equations, that p^2 and q are fractions, as in the above example. These fractions most generally arise from dividing by the co-efficient of x^2 in the reduction of the equation to the required form. When this is the case, we readily discover the quantity by which it is necessary to multiply the terms of q, in order to reduce it to the same denominator with p^2; after which, the numerators may be added together and placed over the common denominator. After this operation, the denominator will be a perfect square, and may be brought from under the radical sign, and will become a divisor of the square root of the numerator.

To apply these principles in reducing the radical part of the values of x, in the last example, we have

$$\sqrt{\frac{360}{22} + \left(\frac{1}{22}\right)^2} = \sqrt{\frac{360 \times 22}{(22)^2} + \frac{1}{(22)^2}} = \sqrt{\frac{7920 + 1}{(22)^2}}$$

$$= \frac{1}{22}\sqrt{7921} = \frac{89}{22};$$

and therefore, the two values of x become,

$$x = -\frac{1}{22} + \frac{89}{22} = \frac{88}{22} = 4;$$

and $$x = -\frac{1}{22} - \frac{89}{2} = -\frac{90}{22} = -\frac{45}{11};$$

either of which values being substituted for x in the given equation, will satisfy it.

3. What are the values of x in the equation

$$ax^2 - ac = cx - bx^2.$$

We have, by transposing and reducing,

$$(a+b)\,x^2 - cx = ac\,;$$

hence, $$x^2 - \frac{c}{a+b}x = \frac{ac}{a+b},$$

and consequently,

$$x = +\frac{c}{2(a+b)} + \sqrt{\frac{ac}{a+b} + \frac{c^2}{4(a+b)^2}},$$

and, $$x = +\frac{c}{2(a+b)} - \sqrt{\frac{ac}{a+b} + \frac{c^2}{4(a+b)^2}}.$$

If now, we multiply both terms of q by $4(a+b)$, it will be reduced to a common denominator with p^2, and we shall have

$$\sqrt{\frac{ac}{a+b} + \frac{c^2}{4(a+b)^2}} = \sqrt{\frac{4a^2c + 4abc + c^2}{4(a+b)^2}} = \frac{\sqrt{4a^2c + 4abc + c^2}}{2(a+b)};$$

hence, $$x = \frac{c \pm \sqrt{4a^2c + 4abc + c^2}}{2(a+b)}.$$

4. What are the values of x, in the equations,

$$6x^2 - 37x = -57.$$

By reducing to the required form, we have

$$x^2 - \frac{37}{6}x = -\frac{57}{6};$$

hence, $$x = +\frac{37}{12} \pm \sqrt{-\frac{57}{6} + \left(\frac{37}{12}\right)^2}$$

We observe, that if we multiply both terms of q by 2, and then by 12, that q and p^2 will have the same denominator; hence,

$$x = +\frac{37}{12} \pm \sqrt{\frac{-114 \times 12}{(12)^2} + \frac{(37)^2}{(12)^2}}.$$

But, $114 \times 12 = 1368$; and $(37)^2 = 1369$;

hence, $$x = +\frac{37}{12} \pm \sqrt{\frac{-1368 + 1369}{(12)^2}} = +\frac{37}{12} \pm \frac{1}{12};$$

hence, $$x = +\frac{37}{12} + \frac{1}{12} = \frac{38}{12} = \frac{19}{6},$$

and $$x = +\frac{37}{12} - \frac{1}{12} = \frac{36}{12} = 3.$$

5. What are the values of x in the equation

$$4a^2 - 2x^2 + 2ax = 18ab - 18b^2.$$

In this equation, the term which contains the second power of the unknown quantity is negative; and since that term already stands in the first member of the equation, it can only be rendered positive by changing the sign of every term of the equation. Doing this, transposing, and dividing by 2, we have

$$x^2 - ax = 2a^2 - 9ab + 9b^2;$$

whence, $$x = \frac{a}{2} \pm \sqrt{2a^2 - 9ab + 9b^2 + \frac{a^2}{4}}$$

$$= \frac{a}{2} \pm \sqrt{\frac{9a^2}{4} - 9ab + 9b^2};$$

and the root of the radical part is equal to $\frac{3a}{2} - 3b$. Hence,

$$x = \frac{a}{2} \pm \left(\frac{3a}{2} - 3b\right); \quad \text{hence,} \quad \begin{cases} x = 2a - 3b. \\ x = -a + 3b. \end{cases}$$

EXAMPLES.

1. Given, $\frac{x}{3} - 4 - x^2 - 2x - \frac{4}{5}x^2 = 45 - 3x^2$, to find x.

Ans. $\begin{cases} x = 7.12 \\ x = -5.73 \end{cases}$ to within 0.01.

2. Given, $x^2 - 8x + 10 = 19$, to find x.

Ans. $x = 9$, and $x = -1$.

3. Given, $x^2 - x - 40 = 170$, to find x.

Ans. $x = 15$, and $x = -14$.

4. Given, $3x^2 + 2x - 9 = 76$, to find x.

Ans. $x = 5$, and $x = -5\frac{2}{3}$.

5. Given, $\frac{1}{2}x^2 - \frac{1}{3}x + 7\frac{3}{8} = 8$, to find x.

Ans. $x = 1\frac{1}{2}$, and $x = -\frac{5}{6}$.

6. Given, $mx^2 - 2mx\sqrt{n} = nx^2 - mn$, to find x.

Ans. $x = \frac{\sqrt{mn}}{\sqrt{m} - \sqrt{n}}$, $x = \frac{\sqrt{mn}}{\sqrt{m} + \sqrt{n}}$.

7. Given $\frac{90}{x} - \frac{90}{x+1} - \frac{27}{x+2} = 0$, to find x.

Ans. $x = 4, \quad x = -\frac{5}{3}$.

8. Given $ax^2 - \frac{ac}{a+b} = cx - bx^2$, to find x.

Ans. $x = \frac{c \pm \sqrt{c^2 + 4ac}}{2(a+b)}$.

9 Given $abx^2 + \frac{3a^2}{c}x = \frac{6a^2 + ab - 2b^2}{c^2} - \frac{b^2x}{c}$, to find x.

Ans. $x = \frac{2a - b}{ac}, \quad x = -\frac{3a + 2b}{bc}$.

10. Given $a^2 + b^2 - 2bx + x^2 = \frac{m^2x^2}{n^2}$, to find x.

Ans. $x = \frac{n}{n^2 - m^2}(bn \pm \sqrt{a^2m^2 + b^2m^2 - a^2n^2})$.

QUESTIONS.

1. To find a number such, that twice its square added to three times the number, shall be equal to 65.

Let x represent the number. Then,

$$2x^2 + 3x = 65,$$

whence, $$x = -\frac{3}{4} \pm \sqrt{\frac{65}{2} + \frac{9}{16}} = -\frac{3}{4} \pm \frac{23}{4};$$

therefore,

$$x = -\frac{3}{4} + \frac{23}{4} = 5, \quad \text{and} \quad x = -\frac{3}{4} - \frac{23}{4} = -\frac{13}{2}.$$

Both these values will satisfy the question, understood in its algebraic sense. We have,

$$2 \times (5)^2 + 3 \times 5 = 2 \times 25 + 15 = 65;$$

and $$2\left(-\frac{13}{2}\right)^2 + 3 \times -\frac{13}{2} = \frac{169}{2} - \frac{39}{2} = \frac{130}{2} = 65.$$

Suppose we had stated the question thus:—To find a number such, that three times the number subtracted from twice its square, shall give a product equal to 65.

remainder

If we denote the number by x we shall have

$$2x^2 - 3x = 65;$$

whence, $$x = \frac{3}{4} \pm \sqrt{\frac{65}{2} + \frac{9}{16}} = \frac{3}{4} \pm \frac{23}{4};$$

therefore, $$x = \frac{3}{4} + \frac{23}{4} = \frac{13}{2}, \text{ and } x = \frac{3}{4} - \frac{23}{4} = -5,$$

values which differ from those found before, only in their signs.

If the last enunciation be understood in its algebraic sense, the -5 equally with the $+\frac{13}{2}$ will satisfy both the enunciation and the equation. It is true that the second term $-3x$ will be *added* to the first term; for, the subtraction of 3 times -5, will give $+15$.

2. A person purchased a number of yards of cloth for 240 cents. If he had received 3 yards less, for the same sum, it would have cost him 4 cents more per yard. How many yards did he purchase?

Let $x =$ the number of yards purchased.

Then, the price per yard will be expressed by $\frac{240}{x}$.

If, for 240 cents, he had received 3 yards less, that is, $x - 3$ yards, the price per yard, under this hypothesis, would have been represented by $\frac{240}{x-3}$. But, by the enunciation this last cost would exceed the first, by 4 cents. Therefore, we have the equation

$$\frac{240}{x-3} - \frac{240}{x} = 4;$$

whence, by reducing, $$x^2 - 3x = 180,$$

and $$x = \frac{3}{2} \pm \sqrt{\frac{9}{4} + 180} = \frac{3 \pm 27}{2};$$

therefore, $$x = 15, \text{ and } x = -12.$$

The value $x = 15$ satisfies the enunciation understood in its arithmetical sense; for, 15 yards for 240 cents, gives $\frac{240}{15}$, or 16 cents for the price of one yard, and 12 yards for 240 cents, gives 20 cents for the price of one yard, which exceeds 16 by 4

The -12 will satisfy the question in its algebraic sense, and considered without reference to its sign, will be the answer to the following arithmetical question:—*A person purchased a number of yards of cloth for* 240 *cents: if he had paid the same sum for* 3 *yards more, it would have cost him* 4 *cents less per yard. How many yards did he purchase?*

REMARK.—In the solution of a problem, both roots of the equation will satisfy the enunciation, understood in its algebraic sense. If the enunciation, considered arithmetically, admits of a double interpretation, when translated into the language of Algebra, the solution of the equation will make known the fact: and hence, while one root resolves the question in its arithmetical sense, the other resolves another similar question also in its arithmetical sense; and both questions will be stated by equations of the same general form, having equal numerical roots with contrary signs.

3. A man bought a horse, which he sold for 24 dollars. At the sale, he lost as much per cent. on the price of his purchase, as the horse cost him. What did he pay for the horse?

Let x denote the number of dollars that he paid for the horse: then, $x - 24$ will express the loss he sustained. But as he lost x per cent. by the sale, he must have lost $\frac{x}{100}$ upon each dollar, and upon x dollars he loses a sum denoted by $\frac{x^2}{100}$; we have then the equation

$$\frac{x^2}{100} = x - 24, \quad \text{whence} \quad x^2 - 100x = -2400;$$

and $$x = 50 \pm \sqrt{-2400 + 2500} = 50 \pm 10.$$

Therefore, $$x = 60 \quad \text{and} \quad x = 40.$$

Both of these values satisfy the question.

For, in the first place, suppose the man gave 60 dollars for the horse and sold him for 24, he then loses 36 dollars. But, from the enunciation he should lose 60 *per cent.* of 60, that is, $\frac{60}{100}$ of $60 = \frac{60 \times 60}{100} = 36$; therefore 60 satisfies the enunciation.

If he pays 40 dollars for the horse, he loses 16 by the sale;

for, he should lose 40 *per cent.* of 40, or $40 \times \frac{40}{100} = 16$; therefore 40 verifies the enunciation.

4. A grazier bought as many sheep as cost him £60, and after reserving 15 out of the number, he sold the remainder for £54, and gained 2*s*. a head on those he sold: how many did he buy? *Ans.* 75.

5. A merchant bought cloth for which he paid £33 15*s*., which he sold again at £2 8*s*. per piece, and gained by the bargain as much as one piece cost him: how many pieces did he buy? *Ans.* 15.

6. What number is that, which, being divided by the product of its digits, the quotient is 3; and if 18 be added to it, the digits will be inverted? $(10-x)x=21$ *Ans.* 24.

7. To find a number such that if you subtract it from 10, and multiply the remainder by the number itself, the product shall be 21. *Ans.* 7 or 3.

8. Two persons, A and B, departed from different places at the same time, and travelled toward each other. On meeting, it appeared that A had travelled 18 miles more than B; and that A could have gone B's journey in $15\frac{3}{4}$ days, but B would have been 28 days in performing A's journey. How far did each travel?

Ans. A 72 miles. B 54 miles.

9. A company at a tavern had £8 15*s*. to pay for their reckoning; but before the bill was settled, two of them left the room, and then those who remained had 10*s*. apiece more to pay than before: how many were there in the company? *Ans.* 7.

10. What two numbers are those whose difference is 15, and of which the cube of the lesser is equal to half their product? *Ans.* 3 and 18.

11. Two partners, A and B, gained $140 in trade: A's money was 3 months in trade, and his gain was $60 less than his stock; B's money was $50 more than A's, and was in trade 5 months: what was A's stock? *Ans.* $100.

12. Two persons, A and B, start from two different points and travel toward each other. When they meet, it appears that A has travelled 30 miles more than B. It also appears that it will take A

4 days to travel the road that B had come, and B 9 days to travel the road that A had come. What was their distance apart when they set out? *Ans.* 150 miles.

Discussion of Equations of the Second Degree.

141. Thus far, we have only resolved particular problems involving equations of the second degree, and in which the known quantities were expressed by particular numbers.

We propose now, to explain the general properties of these equations, and to examine the results which flow from all the suppositions that may be made on the values and signs of the known quantities which enter into them.

142. It has been shown (Art. 140), that every complete equation of the second degree can be reduced to the form

$$x^2 + 2px = q \qquad (1),$$

in which p and q are numerical or algebraic quantities, whole numbers, or fractions, and their signs plus or minus.

If we make the first member a perfect square, by adding p^2 to both members, we have

$$x^2 + 2px + p^2 = q + p^2,$$

which may be put under the form

$$(x + p)^2 = q + p^2.$$

Whatever may be the value of $q + p^2$, its square root may be represented by m, and the equation put under the form

$$(x + p)^2 = m^2, \text{ and consequently, } (x + p)^2 - m^2 = 0.$$

But, as the first member of the last equation is the difference between two squares, it may be put under the form

$$(x + p - m)(x + p + m) = 0, \qquad (2)$$

in which the first member is the product of two factors, and the second 0. Now we can make the product equal to 0, and consequently satisfy equation (2), only in two different ways: viz., making

$$x + p - m = 0, \text{ whence, } x = -p + m,$$

or, by making

$$x + p + m = 0, \text{ whence, } x = -p - m;$$

and by substituting for m its value, we have

$$x = -p + \sqrt{q + p^2}, \text{ and } x = -p - \sqrt{q + p^2}.$$

Now, either of these values being substituted for x in its corresponding factor of equation (2), will satisfy that equation, and consequently, will satisfy equation (1), from which it was derived. Hence we conclude,

1st. *That every equation of the second degree has two roots, and only two.*

2d. *That every equation of the second degree may be decomposed into two binomial factors of the first degree with respect to* x, *having* x *for a first term, and the two roots, taken with their signs changed, for the second terms.*

For example, the equation

$$x^2 + 3x - 28 = 0$$

being resolved gives $x = 4$ and $x = -7$; either of which values will satisfy the equation. We also have

$$(x - 4)(x + 7) = x^2 + 3x - 28.$$

If the roots of an equation are known, we readily form the binomial factors and the equation.

1. What are the factors, and what is the equation, of which the roots are 8 and -9?

$$x - 8 \text{ and } x + 9$$

are the binomial factors, and

$$(x - 8)(x + 9) = x^2 + x - 72 = 0$$

is the equation.

2. What are the factors, and what is the equation, of which the roots are -1 and $+1$.

$$(x + 1)(x - 1) = x^2 - 1 = 0.$$

3. What are the factors and what is the equation, whose roots are

$$\frac{7 + \sqrt{-1039}}{16} \text{ and } \frac{7 - \sqrt{-1039}}{16}.$$

$$\left(x - \frac{7 + \sqrt{-1039}}{16}\right) \times \left(x - \frac{7 - \sqrt{-1039}}{16}\right)$$

$$= 8x^2 - 7x + 34 = 0.$$

143. If we designate the two roots of any equation by x' and x'', we shall have

$$x' = -p + \sqrt{q + p^2}, \text{ and } x'' = -p - \sqrt{q + p^2};$$

by adding the roots, we obtain,

$$x' + x'' = -2p;$$

and by multiplying them together,

$$x'x'' = -q. \quad \text{Hence,}$$

1st. *The algebraic sum of the two roots is equal to the co-efficient of the second term of the equation, taken with a contrary sign.*

2d. *The product of the two roots is equal to the absolute term, taken also with a contrary sign.*

144. Thus far, we have regarded p and q as algebraic quantities, without considering the essential sign of either, nor have we at all regarded their relative values

If we first suppose p and q to be both essentially positive, then to become negative in succession, and after that, both to become negative together, we shall have all the combinations of signs which can arise; and the complete equation of the second degree will, therefore, always be expressed under one of the four following forms:—

$$x^2 + 2px = q \quad (1),$$
$$x^2 - 2px = q \quad (2),$$
$$x^2 + 2px = -q \quad (3),$$
$$x^2 - 2px = -q \quad (4).$$

These equations being resolved, give

$$x = -p \pm \sqrt{q + p^2} \quad (1),$$
$$x = +p \pm \sqrt{q + p^2} \quad (2),$$
$$x = -p \pm \sqrt{-q + p^2} \quad (3),$$
$$x = +p \pm \sqrt{-q + p^2} \quad (4).$$

In order that the value of x, in these equations, may be found either exactly or approximatively, it is necessary that the quantity under the radical sign be positive (Art. 126).

Now, p^2 being necessarily positive, whatever may be the sign of p, it follows, that in the *first* and *second* forms all the values

of x will be real. They will be determined exactly, when the quantity $q + p^2$ is a perfect square, and approximatively, when it is not so.

Since q and p^2 are both positive, the numerical value of the radical expression $\pm\sqrt{q + p^2}$ will be greater than p, and hence the second member of the equation will have the same sign as the radical. Therefore, in the first form, *the first root of the equation will be positive, and the second root negative.*

The positive root will, in general, as already observed, alone satisfy the problem understood in its arithmetical sense; the negative value, answering to a similar problem, differing from the first only in this; that a certain quantity which is regarded as additive in the one, is subtractive in the other, and the reverse.

In the second form, the first value of x is positive, and the second negative, the positive value being the greater.

In the third and fourth forms, the values of x will be imaginary when $q > p^2$, and *real* when $q < p^2$.

And since $\sqrt{-q + p^2} < p$, it follows that the real values of x will both be negative in the third form, and both positive in the fourth.

145. The last properties which have been proved, may be shown from the two properties of an equation of the second degree, demonstrated in Art. 143. They are:

The algebraic sum of the roots is equal to the co-efficient of the second term, taken with a contrary sign, and their product is equal to the absolute term, taken also with a contrary sign.

In the first two forms, q being positive in the second member, it follows that the product of the two roots is negative: hence, *they will have contrary signs.* But in the third and fourth forms, q being negative in the second member, it follows that the product of the two roots will be positive: hence, *they will have like signs*, viz., both *negative* in the third form, where $2p$ is positive, and both positive in the fourth form, where $2p$ is negative.

Moreover, since the sum of the roots is affected with a sign contrary to that of the co-efficient $2p$, it follows that *the negative root will be the greatest in the first form, and the least in the second.*

146. We will now show, that when in the third and fourth forms the roots become imaginary, that is, when $q > p^2$, that the conditions of the question will be incompatible with each other, and therefore, the values of x ought to be imaginary.

Before showing this, it will be necessary to establish a proposition on which it depends: viz.,

If a given number be decomposed into two parts, and those parts multiplied together, the product will be the greatest possible when the parts are equal.

Let a be the number to be decomposed, and d the difference of the parts. Then

$$\frac{a}{2} + \frac{d}{2} = \text{ the greater part (Art. 32).}$$

and $$\frac{a}{2} - \frac{d}{2} = \text{ the less part.}$$

and $$\frac{a^2}{4} - \frac{d^2}{4} = \text{P, their product (Art. 46).}$$

Now it is plain, that P will increase as d diminishes, and that it will be the greatest possible when $d = 0$; that is,

$$\frac{a}{2} \times \frac{a}{2} = \frac{a^2}{4} \text{ is the greatest product.}$$

147. Now, since in the equation

$$x^2 - 2px = -q$$

$2p$ is the sum of the roots, and q their product, it follows that q cannot be greater than p^2. The relations between p and q, therefore, fix a limit to the value of q; and if we assume, arbitrarily, $q > p^2$, we express by the equation a condition which cannot be fulfilled, and this contradiction is made apparent by the values of x becoming imaginary. Analogy would lead us to conclude that,

When the value of the unknown quantity is found to be imaginary, the conditions expressed by the equation are incompatible with each other.

Remark.—Since the roots, in the first and second forms, have contrary signs, the condition that their sum shall be equal to a given number $2p$, does not fix a limit to their product: hence, in those two forms the roots are never imaginary.

148. We shall conclude this discussion by the following re marks :—

1st. If, in the third and fourth forms, we suppose $q = p^2$, the radical part of the two values of x becomes 0, and both the values reduce to $x = \mp p$: *the two roots are then said to be equal.*

In fact, by substituting p^2 for q in the equation, it becomes

$$x^2 \pm 2px = -p^2, \quad \text{whence}$$

$$x^2 \pm 2px + p^2 = 0, \quad \text{that is,} \quad (x \pm p)^2 = 0.$$

Under this supposition, the first member becomes the *product of two equal factors.* Hence, the roots of the equation are equal, since the two factors being placed equal to zero, give the same value for x.

2d. If, in the general equation,

$$x^2 + 2px = q,$$

we suppose $q = 0$, the two values of x reduce to

$$x = -p + p = 0, \quad \text{and} \quad x = -p - p = -2p.$$

Indeed, the equation is then of the form

$$x^2 + 2px = 0, \quad \text{or} \quad x(x + 2p) = 0,$$

which can only be satisfied, either by making

$$x = 0, \quad \text{or} \quad x + 2p = 0;$$

whence, $\quad x = 0, \quad \text{and} \quad x = -2p;$

that is, one of the roots is 0, and the other the co-efficient of x, taken with a contrary sign.

3d. If, in the general equation

$$x^2 \pm 2px = \pm q,$$

we suppose $2p = 0$, there will result

$$x^2 = \pm q, \quad \text{whence,} \quad x = \pm\sqrt{\pm q};$$

that is, in this case *the two values of* x *are equal, and have contrary signs, real in the first and second forms, and imaginary in the third and fourth.*

The equation then belongs to the class of equations involving two terms, treated of in Art. 139.

4th. Suppose we have at the same time $p = 0$, $q = 0$; the equation reduces to $x^2 = 0$, and gives two values of x, equal to 0.

149. There remains a singular case to be examined, which is often met with in the resolution of problems involving equations of the second degree.

To discuss it, take the equation

$$ax^2 + bx = c,$$

which gives $$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}.$$

Suppose, now, that from a particular hypothesis made upon the given quantities of the question, we have $a = 0$; the expression for x becomes

$$x = \frac{-b \pm b}{0}, \quad \text{whence,} \quad \begin{cases} x = \dfrac{0}{0}, \\ x = -\dfrac{2b}{0}, \end{cases}$$

Let us first interpret the first root of $x = \dfrac{0}{0}$.

By multiplying the numerator and denominator of the second member of the equation

$$x = \frac{-b + \sqrt{b^2 + 4ac}}{2a} \quad \text{by} \quad -b - \sqrt{b^2 + 4ac}$$

we obtain

$$x = \frac{b^2 - (b^2 + 4ac)}{2a(-b - \sqrt{b^2 + 4ac})} = \frac{-4ac}{2a(-b - \sqrt{b^2 + 4ac})},$$

hence, $$x = \frac{-2c}{-b - \sqrt{b^2 + 4ac}}, \text{ by dividing by } 2a.$$

and consequently, $$x = \frac{c}{b}, \text{ by making } a = 0.$$

Hence we see that the apparent indetermination arises from a common factor in the numerator and denominator.

In regard to the second root

$$x = -\frac{2b}{0},$$

we see that it is presented under the form of infinity. By making $a = 0$, in the equation

$$ax^2 + bx = c,$$

it reduces to an equation of the first degree,

$$bx = c.$$

It is therefore *impossible* that it can have *two roots;* and hence, such a supposition gives one of the values of x infinite.

We have already seen (Art. 147), that *imaginary values* of the unknown quantity indicate the introduction, into the equation, of contradictory conditions. By considering the above discussion, and that of Art. 110, we would conclude, that a result which is infinite, *indicates the introduction into the equation of a condition that is absolutely impossible.*

If we had at the same time

$$a = 0, \quad b = 0, \quad c = 0,$$

the proposed equation would be altogether indeterminate. This is the only case of indetermination that the equation of the second degree presents.

We are now going to apply the principles of this general discussion to a problem which will give rise to most of the circumstances that are commonly met with in problems involving equations of the second degree.

Problem of the Lights.

150. Find upon the line which joins two lights, A and B, of different intensities, the point which is equally illuminated; admitting the following principle of physics, viz: *The intensity of the same light at two different distances, is in the inverse ratio of the squares of these distances.*

Let the distance AB, between the two lights, be expressed by c; the intensity of the light A, at the units distance, by a; that of the light B, at the same distance, by b. Suppose C to be the equally-illuminated point, and make $AC = x$, whence $BC = c - x$.

By the principle we have assumed, the intensity of A, at the *unity* of distance, being a, its intensity at the distances 2, 3, 4, &c., will be $\frac{a}{4}$, $\frac{a}{9}$, $\frac{a}{16}$, &c.; hence, at the distance x it will be expressed by $\frac{a}{x^2}$. In like manner, the intensity of B at the distance $c - x$, is $\frac{b}{(c-x)^2}$; but, by the conditions, these two

C'' — A — C — B — C'

intensities are equal to each other, and therefore we have the equation

$$\frac{a}{x^2} = \frac{b}{(c-x)^2};$$

which can be put under the form

$$\frac{(c-x)^2}{x^2} = \frac{b}{a}.$$

Hence, $$\frac{c-x}{x} = \frac{\pm\sqrt{b}}{\sqrt{a}};$$ whence,

1st root is, $x = \dfrac{c\sqrt{a}}{\sqrt{a}+\sqrt{b}}$, which gives, $c - x = \dfrac{c\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.

2d root is, $x = \dfrac{c\sqrt{a}}{\sqrt{a}-\sqrt{b}}$, which gives, $c - x = \dfrac{-c\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.

1st. Suppose $a > b$.

The first value of x is positive; and since

$$\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} < 1,$$

it will be less than c, and consequently, the required point C, will be situated between the points A and B. We see moreover that the point will be nearer B than A; for, since $a > b$, we have

$$\sqrt{a} + \sqrt{a} \quad \text{or,} \quad 2\sqrt{a} > (\sqrt{a} + \sqrt{b}), \quad \text{whence,}$$

$$\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} > \frac{1}{2}; \quad \text{and consequently,} \quad \frac{c\sqrt{a}}{\sqrt{a}+\sqrt{b}} > \frac{c}{2}.$$

Indeed, this ought to be the case, since the intensity of A was supposed greater than that of B.

The corresponding value of $c - x$, as may be easily shown, is also positive, and less than one half of c; that is,

$$\frac{c\sqrt{b}}{\sqrt{a}+\sqrt{b}} < \frac{c}{2}.$$

The second value of x is also positive; but since,

$$\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}} > 1,$$

it will be greater than c; and consequently, the required point will be at some point C', on the prolongation of AB, and at the right of the two lights.

We may, in fact, conceive that since the two lights exert their illuminating power in every direction, there should be upon the prolongation of AB, another point equally illuminated; but this point must be nearest that light whose intensity is the least.

We can easily explain, why these two values are connected by the same equation. If, instead of taking AC for the unknown quantity x, we had taken AC', there would have resulted $BC' = x - c$; and the equation

$$\frac{a}{x^2} = \frac{b}{(x-c)^2}.$$

Now, as $(x-c)^2$ is identical with $(c-x)^2$, the new equation is identical with that already established, which consequently should have given AC' as well as AC.

And since every equation is but the algebraic enunciation of a problem, it follows that, *when the same equation enunciates several problems, it ought by its different roots to solve them all.*

When the line AC' is represented by the unknown quantity x, both members of the equation

$$c - x = \frac{-c\sqrt{b}}{\sqrt{a}-\sqrt{b}}$$

are negatived, as they ought to be, since $x > c$.

By changing the signs of both members, we have

$$x - c = \frac{c\sqrt{b}}{\sqrt{a}-\sqrt{b}} = BC'.$$

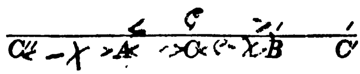
X

2d. Let $a < b$.

This supposition gives a positive value for

$$x = \frac{c\sqrt{a}}{\sqrt{a}+\sqrt{b}};$$

and since $\sqrt{a}+\sqrt{b} > \sqrt{a}+\sqrt{a}$, that is, $> 2\sqrt{a}$



it follows that, $\qquad x < \dfrac{c}{2},$

and consequently, $\qquad c - x > \dfrac{c}{2};$

and therefore, under this hypothesis, the point C, situated between A and B, will be nearer to A than B, as indeed it ought, since the feebler light is at A.

The second value of x, that is,

$$x = \frac{c\sqrt{a}}{\sqrt{a} - \sqrt{b}} = -\frac{c\sqrt{a}}{\sqrt{b} - \sqrt{a}},$$

is essentially negative. How is it to be interpreted?

Let us suppose that we had considered C'', at the left of A, as the point of equal illumination, and that we had represented AC'' by $-x$.

Then, $\qquad BC'' = BA + AC'';$

that is, $\qquad BC'' = c + (-x) = c - x;$

and the equation of the problem would be

$$\frac{a}{(-x)^2} = \frac{b}{(c-x)^2}; \text{ that is, } \frac{a}{x^2} = \frac{b}{(c-x)^2};$$

and therefore, this equation ought to give the point C'' which lies to the left of A, as well as the points C and C' which lie to the right.

It should be observed, that we have regarded $-x$, which represents AC'', as a *mere symbol*, without reference to the essential sign of x. Indeed, the essential sign of the unknown quantity is, in general, *only made known in the final result.*

If it appears, in the final result, that x *itself* is negative, the numerical value of

$$BC'' = c + (-x) \text{ becomes } BC'' = c + x;$$

that is, BC'' will be equal to c plus the numerical value of x, or to c minus its algebraic value. Hence,

$$BC'' = c - \frac{-c\sqrt{a}}{\sqrt{b} - \sqrt{a}} = \frac{c\sqrt{b}}{\sqrt{b} - \sqrt{a}},$$

a quantity which is essentially positive.

3d. Let $a = b$.

Under this supposition, the value of x, and that of $c - x$, for the point C between A and B, both reduce to $\frac{c}{2}$; that is, when the lights are of equal intensity, the point of equal illumination is at the middle of the line AB.

The value of x, and that of $c - x$, for the points C' and C'', which lie on the prolongation of AB, both reduce to

$$\frac{+c\sqrt{a}}{0}, \text{ or, to } \frac{-c\sqrt{b}}{0}, \text{ that is, to infinity;}$$

which indicates, that the conditions of the question are *absolutely impossible*. It is evident, indeed, that they are so; for, when the intensity of the two lights is equal, no part lying on the prolongation of AB could be as much illuminated by the distant as by the nearer light: hence, the supposition of equal illumination, from which the equation of the problem is derived, is *impossible;* and this is shown in the analysis by the corresponding values of the unknown quantity becoming infinite.

4th. Let $a = b$, and $c = 0$.

Under these suppositions, the value of x and of $c - x$, for the point of equal illumination between A and B, both reduce to 0, as indeed they ought to do, since the points A, B, and C, are then united in one.

The value of x, and of $c - x$, for the points C' and C'', reduce to the indeterminate form

$$\frac{0}{0}.$$

Resuming the equation of the problem

$$(a - b)x^2 - 2acx = -c^2a,$$

we see that it becomes, under the above suppositions,

$$0.x^2 - 0.x = 0,$$

which may be satisfied by giving to x any value whatever: hence, *it is a case of indetermination.* Indeed, since the two lights are of the same intensity, and are placed at the same point, *they ought to illuminate equally every point of the straight line.*

5th. Let $c = 0$, and a and b be unequal.

Under this supposition, both values of x, and both values of $c - x$, will reduce to 0; and hence, there is but one point of the line that will be equally illuminated, and that *is the point at which the two lights are placed.*

In this case, the equation of the problem reduces to

$$(a - b)\,x^2 = 0,$$

which gives two values,

$$x = 0, \quad \text{and} \quad x = 0.$$

The preceding discussion presents a striking example of the precision with which the algebraic analysis responds to all the relations which exist between the quantities that enter into the enunciation of a problem.

Examples involving Radicals of the Second Degree.

1. Given, $x + \sqrt{a^2 + x^2} = \dfrac{2a^2}{\sqrt{a^2 + x^2}}$, to find x.

By reducing to entire terms, we have

$$x\sqrt{a^2 + x^2} + a^2 + x^2 = 2a^2,$$

by transposing, $\quad x\sqrt{a^2 + x^2} = a^2 - x^2,$

and by squaring, $\quad a^2x^2 + x^4 = a^4 - 2a^2x^2 + x^4,$

hence, $\quad 3a^2x^2 = a^4.$

and consequently, $\quad x = \pm\sqrt{\dfrac{a^2}{3}}.$

2. Given, $\sqrt{\dfrac{a^2}{x^2} + b^2} - \sqrt{\dfrac{a^2}{x^2} - b^2} = b$, to find x.

By transposing, $\quad \sqrt{\dfrac{a^2}{x^2} + b^2} = \sqrt{\dfrac{a^2}{x^2} - b^2} + b\,;$

and by squaring, $\quad \dfrac{a^2}{x^2} + b^2 = \dfrac{a^2}{x^2} - b^2 + 2b\sqrt{\dfrac{a^2}{x^2} - b^2} + b^2\,;$

hence, $\quad b^2 = 2b\sqrt{\dfrac{a^2}{x^2} - b^2}$, and $b = 2\sqrt{\dfrac{a^2}{x^2} - b^2}\,;$

and by squaring, $\quad b^2 = \dfrac{4a^2}{x^2} - 4b^2\,;$

and hence, $x^2 = \dfrac{4a^2}{5b^2}$, and $x = \pm \dfrac{2a}{b\sqrt{5}}$.

3. Given, $\dfrac{a}{x} + \sqrt{\dfrac{a^2 - x^2}{x^2}} = \dfrac{x}{b}$, to find x.

Ans. $x = \pm \sqrt{2ab - b^2}$.

4. Given, $\sqrt{\dfrac{x+a}{x}} + 2\sqrt{\dfrac{a}{x+a}} = b^2\sqrt{\dfrac{x}{x+a}}$, to find x.

Ans. $x = \dfrac{a}{(b \mp 1)^2}$.

5. Given, $\dfrac{a - \sqrt{a^2 - x^2}}{a + \sqrt{a^2 - x^2}} = b$, to find x.

Ans. $x = \pm \dfrac{2a\sqrt{b}}{1 + b}$.

6. Given, $\dfrac{\sqrt{x} + \sqrt{x-a}}{\sqrt{x} - \sqrt{x-a}} = \dfrac{n^2 a}{x - a}$, to find x.

Ans. $x = \dfrac{a(1 \pm n)^2}{1 \pm 2n}$.

7. Given, $\dfrac{\sqrt{a+x}}{\sqrt{x}} + \dfrac{\sqrt{a-x}}{\sqrt{x}} = \sqrt{\dfrac{x}{b}}$, to find x.

Ans. $x = \pm 2\sqrt{ab - b^2}$.

8. Given, $\dfrac{a + x + \sqrt{2ax + x^2}}{a + x} = b$, to find x.

Ans. $x = \dfrac{\pm a(1 \mp \sqrt{2b - b^2})}{\sqrt{2b - b^2}}$.

Of Trinomial Equations.

151. Every equation which can be reduced to the form

$$x^m + 2px^n = q,$$

in which m and n are positive whole numbers, and $2p$ and q, known quantities, is called *a trinomial equation.*

Hence, a trinomial equation contains three kinds of terms: viz., terms which contain the unknown quantity affected with two different exponents, and one or more known terms.

If we suppose $m = 2$ and $n = 1$, the equation becomes

$$x^2 + 2px = q,$$

a trinomial equation of the second degree.

152. The resolution of trinomial equations of the second degree, has already been explained, and the methods which were pursued are, with some slight modifications, applicable to all trinomial equations in which $m = 2n$, that is, to all equations of the form

$$x^{2n} + 2px^n = q.$$

Let us take, as an example, the trinomial equation of the fourth degree,

$$cx^4 - dx^2 - ax^4 + f = 7 + b.$$

We have, $$x^4 + \frac{-d}{c-a}x^2 = \frac{7+b-f}{c-a};$$

and by substituting $2p$ for the co-efficient of x^2, and q for the absolute term, we have

$$x^4 + 2px^2 = q.$$

If now, we make

$$x^2 = y, \quad \text{and consequently,} \quad x = \pm\sqrt{y},$$

we shall have

$$y^2 + 2py = q, \quad \text{and} \quad y = -p \pm \sqrt{q + p^2}:$$

hence, $$x = \pm\sqrt{-p \pm \sqrt{q + p^2}}.$$

We see that the unknown quantity has four values, since each of the signs + and −, which affect the first radical can be combined in succession with each of the signs which affect the second; *but these values taken two and two are numerically equal, and have contrary signs.*

EXAMPLES.

1. Take the equation

$$x^4 - 25x^2 = -144.$$

If we make $x^2 = y$, the equation becomes,

$$y^2 - 25y = -144,$$

which gives, $$y = 16, \quad \text{and} \quad y = 9.$$

Substituting these values, in succession, for y in the equation $x^2 = y$, and there will result,

1st. $x^2 = 16$, which gives $x = +4$ and $x = -4$.

2d. $x^2 = 9$, which gives $x = +3$ and $x = -3$.

Hence, the four values are $+4$, -4, $+3$, and -3.

2. As a second example, take the equation

$$x^4 - 7x^2 = 8.$$

If we make $x^2 = y$, the equation becomes,

$$y^2 - 7y = 8,$$

which gives $y = 8$, and $y = -1$.

Substituting these values, in succession, for y, and we have

1st. $x^2 = 8$, which gives $x = +2\sqrt{2}$, and $x = -2\sqrt{2}$.

2d. $x^2 = -1$, which gives $x = +\sqrt{-1}$, and $x = -\sqrt{-1}$

The last two values of x are imaginary.

3. Let us take the literal equation

$$x^4 - (2bc + 4a^2)\,x^2 = -b^2c^2.$$

By making $x^2 = y$, we have

$$y^2 - (2bc + 4a^2)\,y = -b^2c^2;$$

whence, $y = bc + 2a^2 \pm 2a\sqrt{bc + a^2}$;

and consequently,

$$x = \pm\sqrt{bc + 2a^2 \pm 2a\sqrt{bc + a^2}}.$$

4. Suppose we have,

$$2x - 7\sqrt{x} = 99.$$

If we make $\sqrt{x} = y$, we have $x = y^2$, and hence,

$$2y^2 - 7y = 99;$$

from which we obtain

$$y = 9, \text{ and } y = -\frac{11}{2}.$$

hence, $x = 81$, and $x = \dfrac{121}{4}$.

153. Before resolving the general case of trinomial equations, it may be well to remark that, *the* n*th root of any quantity, is an expression which multiplied by itself* n − 1 *times will produce the given quantity.*

The method of finding the nth root has not yet been explained, but it is sufficient for our present purpose that we are able to indicate it.

Let it be required to find the values of y in the equation

$$y^{2n} + 2py^n = q.$$

If we make $y^n = x$, we have $y^{2n} = x^2$, and hence, the given equation becomes

$$x^2 + 2px = q,$$

and hence, $\quad x = -p \pm \sqrt{q + p^2};$

that is, $\quad y^n = -p \pm \sqrt{q + p^2},$

and $\quad y = \sqrt[n]{-p \pm \sqrt{q + p^2}}.$

If we suppose $n = 2$, the given equation becomes a trinomial equation of the fourth degree, and we have

$$y = \sqrt{-p \pm \sqrt{q + p^2}}.$$

154. The resolution of trinomial equations of the fourth degree, therefore, gives rise to a new species of algebraic operation: viz., the extraction of the square root of a quantity of the form

$$a \pm \sqrt{b},$$

in which a and b are numerical or algebraic quantities.

To illustrate the transformations which may be effected in expressions of this form, let us take the expression $3 \pm \sqrt{5}$.

By squaring it we have

$$(3 \pm \sqrt{5})^2 = 9 \pm 6\sqrt{5} + 5 = 14 \pm 6\sqrt{5}:$$

hence, reciprocally, $\sqrt{14 \pm 6\sqrt{5}} = 3 \pm \sqrt{5}.$

As a second example, we have

$$(\sqrt{7} \pm \sqrt{11})^2 = 7 \pm 2\sqrt{77} + 11 = 18 \pm 2\sqrt{77}:$$

hence, reciprocally, $\sqrt{18 \pm 2\sqrt{77}} = \sqrt{7} \pm \sqrt{11}.$

Hence we see, that an expression of the form

$$\sqrt{a \pm \sqrt{b}},$$

may sometimes be reduced to the form

$$a' \pm \sqrt{b'} \quad \text{or} \quad \sqrt{a'} \pm \sqrt{b'};$$

and when this transformation is possible, it is advantageous to effect it, since in this case we have only to extract two simple square roots; whereas, the expression

$$\sqrt{a \pm \sqrt{b}}$$

requires the extraction of the square root of the square root.

155. If we represent two indeterminate quantities by p and q, we can always attribute to them such values as to satisfy the equations

$$p + q = \sqrt{a + \sqrt{b}} \quad - \quad - \quad - \quad (1),$$

and

$$p - q = \sqrt{a - \sqrt{b}} \quad - \quad - \quad - \quad (2).$$

These equations being multiplied together, give

$$p^2 - q^2 = \sqrt{a^2 - b} \quad - \quad - \quad - \quad - \quad (3).$$

Now, if p and q are irrational monomials involving only *single radicals of the second degree*, or, if only one is irrational, it follows that p^2 and q^2 will be rational; in which case, $p^2 - q^2$, or its value, $\sqrt{a^2 - b}$, is necessarily a rational quantity, and consequently, $a^2 - b$ is a perfect square.

Under this supposition, a transformation can always be effected that will simplify the expression.

By squaring equations (1) and (2), we have

$$p^2 + 2pq + q^2 = a + \sqrt{b}$$

$$p^2 - 2pq + q^2 = a - \sqrt{b},$$

and by adding member to member,

$$p^2 + q^2 = a \quad - \quad - \quad (4).$$

If we denote the second member of equation (3) by c, we shall have

$$p^2 - q^2 = c \quad - \quad - \quad (5).$$

By adding the two last equations and subtracting equation (5) from (4), we have

$$2p^2 = a + c, \quad \text{and} \quad 2q^2 = a - c.$$

and therefore, $p = \sqrt{\dfrac{a + c}{2}}$, and $q = \sqrt{\dfrac{a - c}{2}}$;

and consequently,

$$\sqrt{a+\sqrt{b}}, \quad \text{or} \quad p+q=\pm\sqrt{\frac{a+c}{2}}\pm\sqrt{\frac{a-c}{2}},$$

$$\sqrt{a-\sqrt{b}}, \quad \text{or} \quad p-q=\pm\sqrt{\frac{a+c}{2}}\mp\sqrt{\frac{a-c}{2}};$$

hence,

$$\sqrt{a+\sqrt{b}}=\pm\left(\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}}\right) \quad . \quad . \quad (6),$$

and $$\sqrt{a-\sqrt{b}}=\pm\left(\sqrt{\frac{a+c}{2}}-\sqrt{\frac{a-c}{2}}\right) \quad . \quad . \quad (7).$$

These two formulas can be verified; for by squaring both members of the first, it becomes

$$a+\sqrt{b}=\frac{a+c}{2}+\frac{a-c}{2}+2\sqrt{\frac{a^2-c^2}{4}}=a+\sqrt{a^2-c^2};$$

but, $\sqrt{a^2-b}=c$, gives $c^2=a^2-b$.

Hence, $a+\sqrt{b}=a+\sqrt{a^2-a^2+b}=a+\sqrt{b}.$

The second formula can be verified in the same manner.

REMARK.—156. Formulas (6) and (7) have been deduced without reference to any particular value of c; and hence, they are equally *true* whether c be *rational* or *irrational*. If, however, c is irrational, they will not simplify the given expression, for each will contain a double radical. Therefore, in general, this transformation is not used, unless a^2-b is a perfect square.

EXAMPLES.

1. Reduce $\sqrt{94+42\sqrt{5}}=\sqrt{94+\sqrt{8820}}$, to its simplest form. We have $a=94$, $b=8820$,

whence, $c=\sqrt{a^2-b}=\sqrt{8836-8820}=4$,

a rational quantity; therefore, formula (6) is applicable to this case, and we have

$$\sqrt{94+42\sqrt{5}}=\pm\left(\sqrt{\frac{94+4}{2}}+\sqrt{\frac{94-4}{2}}\right),$$

or, reducing, $=\pm(\sqrt{49}+\sqrt{45});$

therefore, $\sqrt{94 + 42\sqrt{5}} = \pm (7 + 3\sqrt{5})$.

Indeed,

$$(7 + 3\sqrt{5})^2 = 49 + 45 + 42\sqrt{5} = 94 + 42\sqrt{5}).$$

2. Reduce $\sqrt{np + 2m^2 - 2m\sqrt{np + m^2}}$, to its simplest form. We have

$$a = np + 2m^2, \quad \text{and} \quad b = 4m^2(np + m^2),$$

$$a^2 - b = n^2p^2, \quad \text{and} \quad c = \sqrt{a^2 - b} = np;$$

and therefore, formula (7) is applicable. It gives,

$$\pm \left(\sqrt{\frac{np + 2m^2 + np}{2}} - \sqrt{\frac{np + 2m^2 - np}{2}}\right),$$

and, reducing, $\pm (\sqrt{np + m^2} - m)$.

Indeed, $(\sqrt{np + m^2} - m)^2 = np + 2m^2 - 2m\sqrt{np + m^2}$.

3. Reduce to its simplest form,

$$\sqrt{16 + 30\sqrt{-1}} + \sqrt{16 - 30\sqrt{-1}}.$$

By applying the formulas, we find

$$\sqrt{16 + 30\sqrt{-1}} = 5 + 3\sqrt{-1},$$

and $$\sqrt{16 - 30\sqrt{-1}} = 5 - 3\sqrt{-1}:$$

hence, $$\sqrt{16 + 30\sqrt{-1}} + \sqrt{16 - 30\sqrt{-1}} = 10.$$

This last example shows very clearly the utility of the general problem; because it proves that *imaginary expressions* combined together, may produce *real*, and even *rational results*.

4. Reduce to its simplest form,

$$\sqrt{28 + 10\sqrt{3}}. \qquad \textit{Ans. } 5 + \sqrt{3}.$$

5. Reduce to its simplest form,

$$\sqrt{1 + 4\sqrt{-3}}. \qquad \textit{Ans. } 2 + \sqrt{-3}.$$

6. Reduce to its simplest form,

$$\sqrt{bc + 2b\sqrt{bc - b^2}} + \sqrt{bc - 2b\sqrt{bc - b^2}}.$$

Ans. $\pm 2b$.

7. Reduce to its simplest form,

$$\sqrt{ab + 4c^2 - d^2 - 2\sqrt{4abc^2 - abd^2}}.$$

Ans. $\sqrt{ab} - \sqrt{4c^2 - d^2}.$

Equations of the Second Degree involving two or more Unknown Quantities.

157. An equation involving two or more unknown quantities, is said to be of the *second degree*, when *the greatest sum of the exponents of the unknown quantities, in any term, is equal to* 2. Thus,

$$3x^2 - 4x + y^2 - xy - dy + 6 = 0, \quad 7xy - 4x + y = 0,$$

are equations of the second degree.

Hence, every general equation of the second degree, involving two unknown quantities, may be reduced to the form

$$ay^2 + bxy + cx^2 + dy + fx + g = 0,$$

a, b, c, &c., representing known quantities, either numerical or algebraic.

Take the two equations

$$a\,y^2 + b\,xy + c\,x^2 + d\,y + f\,x + g = 0,$$
$$a'y^2 + b'xy + c'x^2 + d'y + f'x + g' = 0.$$

Arranging them with reference to x, they become

$$c\,x^2 + (b\,y + f\,)\,x + a\,y^2 + d\,y + g = 0,$$
$$c'x^2 + (b'y + f')\,x + a'y^2 + d'y + g' = 0;$$

from which we may eliminate x^2, after having made its co-efficient the same in both equations.

By multiplying the first equation by c', and the second by c, they become

$$cc'x^2 + (b\,y + f\,)\,c'x + (a\,y^2 + d\,y + g\,)\,c' = 0,$$
$$cc'x^2 + (b'y + f'\,)\,c\,x + (a'y^2 + d'y + g')\,c = 0.$$

Subtracting one from the other, we have

$$[(bc' - cb')\,y + fc' - cf']\,x + (ac' - ca')\,y^2 + (dc' - cd')\,y + gc' - cg' = 0,$$

which gives

$$x = \frac{(ca' - ac')\,y^2 + (cd' - dc')\,y + cg' - gc'}{(bc' - cb')\,y + fc' - cf'}.$$

This value, being substituted for x in one of the proposed equations, will give *a final equation*, involving y.

But without effecting the substitution, which would lead to a very complicated result, it is easy to perceive that the final equation involving y will be of the fourth degree. For, the numerator of the value of x being of the form $my^2 + ny + p$, its square will be of the fourth degree, and this square forms one of the parts of the result of the substitution.

Therefore, in general, *the resolution of two equations of the second degree, involving two unknown quantities, depends upon that of an equation of the fourth degree, involving one unknown quantity.*

158. The manner of resolving a general equation of the fourth degree, not having been yet explained, we cannot here give a complete theory of this subject. We will, however, indicate some of the particular methods by which equations of the second degree involving two or more unknown quantities, may be resolved by an equation of the second degree involving but one.

1. Find two numbers such, that the sum of the respective products of the first multiplied by a, and the second multiplied by b, shall be equal to $2s$; and the product of the one by the other equal to p.

Let x and y denote the required numbers, and we have

$$ax + by = 2s,$$

and, $$xy = p.$$

From the first

$$y = \frac{2s - ax}{b};$$

whence, by substituting in the second, and reducing,

$$ax^2 - 2sx = -bp.$$

Therefore, $$x = \frac{s}{a} \pm \frac{1}{a}\sqrt{s^2 - abp},$$

and consequently, $$y = \frac{s}{b} \mp \frac{1}{b}\sqrt{s^2 - abp},$$

This problem is susceptible of two direct solutions, because

$$s > \sqrt{s^2 - abp},$$

but in order that the roots may be real, it is necessary that

$$s^2 > \text{ or } = abp.$$

Let $a = b = 1$; the values of x, and y, then reduce to

$$x = s \pm \sqrt{s^2 - p}, \text{ and } y = s \mp \sqrt{s^2 - p};$$

whence we see, that under this supposition, the two values of x are equal to those of y, taken in an inverse order; which shows, that if

$$s + \sqrt{s^2 - p} \text{ represents the value of } x, \quad s - \sqrt{s^2 - p}$$

will represent the corresponding value of y, and reciprocally.

This relation is explained by observing, that, under the last supposition, the given equations become

$$x + y = 2s, \quad \text{and} \quad xy = p;$$

and the question is then reduced to *finding two numbers of which the sum is* 2s, *and their product* p, or in other words, *to divide a number* 2s, *into two such parts, that their product may be equal to a given number* p.

2. Find four numbers in proportion, knowing the sum 2s of their extremes, the sum $2s'$ of the means, and the sum $4c^2$ of their squares.

Let u, x, y, z, denote the four terms of the proportion; the equations of the problem will be

1st condition, - - -	$u + z = 2s,$
2d condition, - - -	$x + y = 2s',$
since they are in proportion,	$uz = xy,$
4th condition, - -	$u^2 + x^2 + y^2 + z^2 = 4c^2.$

At first sight, it may appear difficult to find the values of the unknown quantities, but with the aid of an *unknown auxiliary*, they are easily determined.

Let p be the unknown product of the extremes or means; we shall then have

$$\left\{\begin{array}{r} u + z = 2s, \\ uz = p, \end{array}\right\} \text{ which give, } \left\{\begin{array}{l} u = s + \sqrt{s^2 - p}, \\ z = s - \sqrt{s^2 - p}. \end{array}\right.$$

and

$$\left\{\begin{array}{r} x + y = 2s', \\ xy = p, \end{array}\right\} \text{ which give, } \left\{\begin{array}{l} x = s' + \sqrt{s'^2 - p}, \\ y = s' - \sqrt{s'^2 - p}. \end{array}\right.$$

Hence, we see that the determination of the four unknown quantities depends only upon that of the product p.

Now, by substituting these values of u, x, y, z, in the last of the equations of the problem, it becomes

$$(s+\sqrt{s^2-p})^2+(s-\sqrt{s^2-p})^2+(s'+\sqrt{s'^2-p})^2 + (s'-\sqrt{s'^2-p})^2=4c^2;$$

and by developing and reducing,

$$4s^2+4s'^2-4p=4c^2; \quad \text{hence,} \quad p=s^2+s'^2-c^2.$$

Substituting this value for p, in the expressions for u, x, y, z, we find

$$\begin{cases} u=s+\sqrt{c^2-s'^2}, \\ z=s-\sqrt{c^2-s'^2}, \end{cases} \qquad \begin{cases} x=s'+\sqrt{c^2-s^2}, \\ y=s'-\sqrt{c^2-s^2}. \end{cases}$$

These four numbers evidently form a proportion; for we have

$$uz=(s+\sqrt{c^2-s'^2})(s-\sqrt{c^2-s'^2})=s^2-c^2+s'^2,$$

$$xy=(s'+\sqrt{c^2-s^2})(s'-\sqrt{c^2-s^2})=s'^2-c^2+s^2.$$

REMARK.—This problem shows how much the introduction of an *unknown auxiliary* facilitates the determination of the principal unknown quantities. There are other problems of the same kind, which lead to equations of a degree superior to the second, and yet they may be resolved by the aid of equations of the first and second degrees, by introducing *unknown auxiliaries.*

3. Given the sum of two numbers equal to a, and the sum of their cubes equal to c, to find the numbers

By the conditions $\begin{cases} x+y=a \\ x^3+y^3=c. \end{cases}$

Putting $x=s+z$, and $y=s-z$, we have $a=2s$,

and $\begin{cases} x^3=s^3+3s^2z+3sz^2+z^3 \\ y^3=s^3-3s^2z+3sz^2-z^3: \end{cases}$

hence, by addition, $x^3+y^3=2s^3+6sz^2=c;$

whence, $z^2=\dfrac{c-2s^3}{6s}$, and $z=\pm\sqrt{\dfrac{c-2s^3}{6s}}$,

or, $x=s\pm\sqrt{\dfrac{c-2s^3}{6s}}$; and $y=s\mp\sqrt{\dfrac{c-2s^2}{6s}}$;

and by substituting for s its value,

$$x = \frac{a}{2} \pm \sqrt{\left(\frac{c - \frac{1}{4}a^3}{3a}\right)} = \frac{a}{2} \pm \sqrt{\frac{4c - a^3}{12a}},$$

and $$y = \frac{a}{2} \mp \sqrt{\left(\frac{c - \frac{1}{4}a^3}{3a}\right)} = \frac{a}{2} \mp \sqrt{\frac{4c - a^3}{12a}}.$$

4. Given, $\frac{xy}{\frac{\sqrt{x}}{\sqrt{y}}} = 48$, and $\frac{xy}{\sqrt{x}} = 24$, to find x and y.

Dividing the first equation by the second, we have

$$\frac{\sqrt{x}}{\sqrt{\frac{x}{y}}} = \sqrt{y} = 2, \text{ and hence } y = 4.$$

Whence, from the second equation we have,

$$\frac{4x}{\sqrt{x}} = 4\sqrt{x} = 24,$$

and consequently, $\sqrt{x} = 6$, and $x = 36$.

5. Given, $x + \sqrt{xy} + y = 19$ and $x^2 + xy + y^2 = 133$ } to find x and y.

Dividing the second equation by the first, we have

$$x - \sqrt{xy} + y = 7,$$

but, $x + \sqrt{xy} + y = 19$:

hence, $2x + 2y = 26$ by addition,

or, $x + y = 13$;

and $\sqrt{xy} + 13 = 19$ by substituting in the 1st eq.;

or, $\sqrt{xy} = 6$

and $xy = 36$.

The 2d equation is, $x^2 + xy + y^2 = 133$,

and from the last, $3xy = 108$;

by subtracting $x^2 - 2xy + y^2 = 25$:

hence, $x - y = \pm 5$.

But, $x + y = 13$:

hence, $x = 9$, or 4; and $y = 4$, or 9.

6. Find the values of x and y, in the equations

$$x^2 + 3x + y = 73 - 2xy$$
$$y^2 + 3y + x = 44.$$

By transposition, the first equation becomes,

$$x^2 + 2xy + 3x + y = 73;$$

to which, if the second be added, there results,

$$x^2 + 2xy + y^2 + 4x + 4y = (x + y)^2 + 4(x + y) = 117.$$

If now, in the equation

$$(x + y)^2 + 4(x + y) = 117,$$

we regard $x + y$ as a single unknown quantity, we shall have

$$x + y = -2 \pm \sqrt{117 + 4};$$

hence, $\quad x + y = -2 + 11 = 9,$

and $\quad x + y = -2 - 11 = -13;$

whence, $\quad x = 9 - y, \quad \text{and} \quad x = -13 - y.$

Substituting these values of x in the second equation, we have

$$y^2 + 2y = 35, \quad \text{for} \quad x = 9,$$

and $\quad y^2 + 2y = 57, \quad \text{for} \quad x = -13.$

The first equation gives,

$$y = 5, \quad \text{and} \quad y = -7.$$

and the second,

$$y = -1 + \sqrt{58}, \quad \text{and} \quad y = -1 - \sqrt{58}.$$

The corresponding values of x, are

$$x = 4, \qquad x = 16;$$

$$x = -12 - \sqrt{58}, \quad \text{and} \quad x = -12 + \sqrt{58}.$$

7. Find the values of x and y, in the equations

$$x^2y^2 + xy^2 + xy = 600 - (y + 2)x^2y^3$$
$$x + y^2 = 14 - y.$$

From the first equation, we have

$$x^2y^2 + (y^2 + 2y)x^2y^2 + xy^2 + xy = 600,$$

or, $\quad x^2y^2(1 + y^2 + 2y) + xy(1 + y) = 600,$

or, again, $\quad x^2y^2(1 + y)^2 + xy(1 + y) \qquad = 600;$

which is the form of an equation of the second degree, by regarding $xy(1+y)$ as the unknown quantity. Hence,

$$xy(1+y) = -\tfrac{1}{2} \pm \sqrt{600 + \tfrac{1}{4}} = -\tfrac{1}{2} \pm \sqrt{\frac{2401}{4}};$$

and if we discuss only the roots which belong to the + value of the radical, we have

$$xy(1+y) = -\tfrac{1}{2} + \frac{49}{2} = 24;$$

and hence, $$x = \frac{24}{y+y^2}.$$

Substituting this value of x in the second equation, we have

$$(y^2+y)^2 - 14(y^2+y) = -24;$$

whence, $\quad y^2 + y = 12, \quad$ and $\quad y^2 + y = 2.$

From the first equation, we have

$$y = -\frac{1}{2} \pm \frac{7}{2} = 3, \text{ or } -4;$$

and the corresponding values of x, from the equation

$$x = \frac{24}{y^2+y} = 2.$$

From the second equation, we have

$$y = 1, \text{ and } y = -2;$$

which gives $\quad x = 12.$

8. Given, $x^2y + xy^2 = 6$, and $x^3y^2 + x^2y^3 = 12$, to find x and y.

Ans. $\begin{cases} x = 2 \text{ or } 1, \\ y = 1 \text{ or } 2. \end{cases}$

9. Given, $\begin{Bmatrix} x^2 + x + y = 18 - y^2 \\ xy = 6 \end{Bmatrix}$ to find x and y.

Ans. $\begin{cases} x = 3, \text{ or } 2; \text{ or } -3 \pm \sqrt{3}, \\ y = 2, \text{ or } 3; \text{ or } -3 \mp \sqrt{3}. \end{cases}$

QUESTIONS.

1. There are two numbers whose difference is 15, and half their product is equal to the cube of the lesser number. What are the numbers? *Ans.* 3 and 18.

2. What two numbers are those whose sum multiplied by the greater, is equal to 77; and whose difference, multiplied by the lesser, is equal to 12?

Ans. 4 and 7, or $\frac{3}{2}\sqrt{2}$ and $\frac{11}{2}\sqrt{2}$.

3. To divide 100 into two such parts, that the sum of their square roots may be 14. *Ans.* 64 and 36.

4. It is required to divide the number 24 into two such parts, that their product may be equal to 35 times their difference.

Ans. 10 and 14.

5. The sum of two numbers is 8, and the sum of their cubes is 152. What are the numbers? *Ans.* 3 and 5.

6. The sum of two numbers is 7, and the sum of their 4th powers is 641. What are the numbers? *Ans.* 2 and 5.

7. The sum of two numbers is 6, and the sum of their 5th powers is 1056. What are the numbers? *Ans.* 2 and 4.

8. Two merchants each sold the same kind of stuff: the second sold 3 yards more of it than the first, and together, they received 35 dollars. The first said to the second, "I would have received 24 dollars for your stuff." The other replied, "And I would have received $12\frac{1}{2}$ dollars for yours." How many yards did each of them sell?

$$Ans. \begin{Bmatrix} \text{1st merchant } x = 15 \\ \text{2d - - - } y = 18 \end{Bmatrix} \text{ or } \begin{Bmatrix} x = 5 \\ y = 8. \end{Bmatrix}$$

9. A widow possessed 13,000 dollars, which she divided into two parts, and placed them at interest, in such a manner, that the incomes from them were equal. If she had put out the first portion at the same rate as the second, she would have drawn for this part 360 dollars interest; and if she had placed the second out at the same rate as the first, she would have drawn for it 490 dollars interest. What were the two rates of interest?

Ans. 7 and 6 per cent.

CHAPTER VII.

OF PROPORTIONS AND PROGRESSIONS.

159. Two quantities of the same kind may be compared together in two ways :—

1st. By considering *how much* one is greater or less than the other, which is shown by their difference; and

2d. By considering *how many times* one is greater or less than the other, which is shown by their quotient.

Thus, in comparing the numbers 3 and 12 together with respect to their difference, we find that 12 *exceeds* 3, by 9; and in comparing them together with respect to their quotient, we find that 12 *contains* 3, four times, or that 12 is 4 times as great as 3.

The first of these methods of comparison is called *Arithmetical Proportion;* and the second, *Geometrical Proportion.* Hence,

ARITHMETICAL PROPORTION *considers the relation of quantities to each other, with respect to their difference; and* GEOMETRICAL PROPORTION, *the relation of quantities to each other, with respect to their quotient.*

Of Arithmetical Proportion.

160. If we have four numbers,

2, 4, 8, and 10,

of which the difference between the first and second is equal to the difference between the third and fourth, these numbers are said to be in arithmetical proportion. The first term 2 is called an *antecedent*, and the second term 4, with which it is compared, a *consequent*. The number 8 is also called an antecedent, and the number 10, with which it is compared, a consequent. The

first and fourth terms are called the *extremes;* and the second and third terms, the *means.*

Let a, b, c, and d, denote four quantities in arithmetical proportion; and d the difference between either antecedent and its consequent.

Then, $a - b = d$, and $a = b + d$;

also, $c - d = d$, and $d = c - d$.

By adding the last two equations, we have

$$a + d = b + c: \text{ that is,}$$

If four quantities are in arithmetical proportion, the sum of the two extremes is equal to the sum of the two means.

Arithmetical Progression.

161. When the difference between the first antecedent and consequent is the same as between any two consecutive terms of the proportion, the proportion is called an *arithmetical progression.* Hence, an *arithmetical progression*, or a *progression by differences*, is a succession of terms, each of which is greater or less than the one that precedes it by a constant quantity, which is called the *common difference* of the progression. Thus,

1, 4, 7, 10, 13, 16, 19, 22, 25, . . .

and 60, 56, 52, 48, 44, 40, 36, 32, 28, . . .

are arithmetical progressions. The first is called an *increasing progression*, of which the common difference is 3; and the second, a *decreasing progression*, of which the common difference is 4.

An arithmetical progression, is also called, an *arithmetical series;* and generally,

A series is a succession of terms derived from each other according to some fixed and known law.

Let a, b, c, d, e, f, . . . designate the terms of a progression by differences; it has been agreed to write them thus:

$$a \,.\, b \,.\, c \,.\, d \,.\, e \,.\, f \,.\, g \,.\, h \,.\, i \,.\, k \; . \, . \, .$$

This series is read, a is to b, as b is to c, as c is to d, as d is to e, &c. This is a series of *continued equi-differences*, in which

each term is at the same time a consequent and antecedent, with the exception of the first term, which is only an *antecedent*, and the last, which is only a *consequent*.

162. Let d represent the common difference of the progression

$$a \,.\, b \,.\, c \,.\, e \,.\, f \,.\, g \,.\, h \,.\, k, \text{ \&c.},$$

which we will consider increasing.

From the definition of a progression, it follows that,

$$b = a + d, \quad c = b + d = a + 2d, \quad e = c + d = a + 3d;$$

and, in general, any term of the series, is equal to *the first term plus as many times the common difference as there are preceding terms.*

Thus, let l be any term, and n the number which marks the place of it. Then, the number of preceding terms will be denoted by $n - 1$, and the expression for this *general term*, will be

$$l = a + (n - 1)\,d.$$

That is, *any term is equal to the first term, plus the product of the common difference by the number of preceding terms.*

If we make $n = 1$, we have $l = a$; that is, the series will have but one term.

If we make

$$n = 2, \text{ we have } l = a + d;$$

that is, the series will have two terms, and the second term is equal to the first plus the common difference.

EXAMPLES.

1. If $a = 3$ and $d = 2$, what is the 3d term? *Ans.* 7.

2. If $a = 5$ and $d = 4$, what is the 6th term? *Ans.* 25.

3. If $a = 7$ and $d = 5$, what is the 9th term? *Ans.* 47.

The formula,

$$l = a + (n - 1)\,d,$$

serves to find any term whatever, without determining all those which precede it.

Thus, to find the 50th term of the progression,

$$1 . 4 . 7 . 10 . 13 . 16 . 19, \ldots$$

we have, $l = 1 + 49 \times 3 = 148.$

And for the 60th term of the progression,

$$1 . 5 . 9 . 13 . 17 . 21 . 25, \ldots$$

we have, $l = 1 + 59 \times 4 = 237.$

163. If the progression were a decreasing one, we should have

$$l = a - (n - 1)d.$$

That is, *any term in a decreasing arithmetical progression, is equal to the first term minus the product of the common difference by the number of preceding terms.*

EXAMPLES.

1. The first term of a decreasing progression is 60, and the common difference 3: what is the 20th term?

$l = a - (n - 1)d$ gives $l = 60 - (20 - 1)3 = 60 - 57 = 3.$

2. The first term is 90, the common difference 4: what is the 15th term? *Ans.* 34.

3. The first term is 100, and the common difference 2: what is the 40th term? *Ans.* 22.

164. A progression by differences being given, it is proposed to prove that, *the sum of any two terms, taken at equal distances from the two extremes, is equal to the sum of the two extremes.*

Let $a . b . c . e . f \ldots . i . k . l$, be the proposed progression, and n the number of terms.

We will first observe that, if x denote a term which has p terms before it, reckoning from the first term, and y a term which has p terms before it, reckoning from the last term, we have, from what has been said,

$$x = a + p \times d,$$

and $$y = l - p \times d;$$

whence, by addition, $x + y = a + l.$

Now, to find the sum of all the terms, write the progression below itself, but in an inverse order, viz.,

$$a\,.\,b\,.\,c\,.\,e\,.\,f\,.\,.\,.\,.\,i\,.\,k\,.\,l.$$

$$l\,.\,k\,.\,i\,\,.\,.\,.\,.\,.\,.\,.\,c\,.\,b\,.\,a.$$

Calling S the sum of the terms of the first progression, $2S$ will be the sum of the terms in both progressions, and we shall have

$$2S = (a + l) + (b + k) + (c + i) \ldots + (i + c) + (k + b) + (l + a).$$

And, since all the parts $a + l$, $b + k$, $c + i \ldots$ are equal to each other, and their number equal to n, by which we designate the number of terms in each series, we have

$$2S = (a + l)\,n, \quad \text{or} \quad S = \left(\frac{a + l}{2}\right) n.$$

That is, *the sum of the terms of an arithmetical progression, is equal to half the sum of the two extremes multiplied by the number of terms.*

EXAMPLES.

1. The extremes are 2 and 16, and the number of terms 8: what is the sum of the series?

$$S = \left(\frac{a + l}{2}\right) \times n, \quad \text{gives} \quad S = \frac{2 + 16}{2} \times 8 = 72.$$

2. The extremes are 3 and 27, and the number of terms 12: what is the sum of the series? *Ans.* 180.

3. The extremes are 4 and 20, and the number of terms 10: what is the sum of the series? *Ans.* 120.

4. The extremes are 8 and 80, and the number of terms 10: what is the sum of the series? *Ans.* 440.

X 165. The formulas

$$l = a + (n - 1)\,d \quad \text{and} \quad S = \left(\frac{a + l}{2}\right) \times n,$$

contain five quantities, a, d, n, l, and S, and consequently give rise to the following general problem, viz.: *Any three of these five quantities being given, to determine the other two.*

This general problem gives rise to the ten following cases:—

No.	Given.	Unknown.	Values of the unknown quantities.
1	a, d, n	l, S	$l = a + (n-1)d$; $S = \frac{1}{2}n[2a + (n-1)d]$.
2	a, d, l	n, S	$n = \frac{l-a}{d} + 1$; $S = \frac{(l+a)(l-a+d)}{2d}$.
3	a, d, S	n, l	$n = \frac{d-2a \pm \sqrt{(d-2a)^2+8dS}}{2d}$; $l = a+(n-1)d$.
4	a, n, l	S, d	$S = \frac{1}{2}n(a+l)$; $d = \frac{l-a}{n-1}$.
5	a, n, S	d, l	$d = \frac{2(S-an)}{n(n-1)}$; $l = \frac{2S}{n} - a$.
6	a, l, S	n, d	$n = \frac{2S}{a+l}$; $d = \frac{(l+a)(l-a)}{2S-(l+a)}$.
7	d, n, l	a, S	$a = l-(n-1)d$; $S = \frac{1}{2}n[2l-(n-1)d]$.
8	d, n, S	a, l	$a = \frac{2S-n(n-1)d}{2n}$; $l = \frac{2S+n(n-1)d}{2n}$.
9	d, l, S	n, a	$n = \frac{2l+d \pm \sqrt{(2l+d)^2-8dS}}{2d}$; $a = l-(n-1)d$.
10	n, l, S	a, d	$a = \frac{2S}{n} - l$; $d = \frac{2(nl-S)}{n(n-1)}$.

The solution of these cases presents no difficulty. Cases 3 and 9 give rise to equations of the second degree; but one of the roots will always satisfy the enunciation of the question in its arithmetical sense.

If we resume the formula

$$l = a + (n-1)d,$$

we have, $\qquad a = l - (n-1)d$; that is,

The first term of an increasing arithmetical progression, is equal to any following term, minus the product of the common difference by the number of preceding terms.

From the same formula, we also find

$$d = \frac{l-a}{n-1}; \text{ that is,}$$

In any arithmetical progression, the common difference is equal to the difference between the first and last terms considered, divided by the number of terms less one.

1. Two terms of a progression are 16 and 4, and the number of terms considered is 5: what is the common difference?

The formula

$$d = \frac{l - a}{n - 1} \quad \text{gives} \quad d = \frac{16 - 4}{4} = 3.$$

2. Two terms of a progression are 22 and 4, and the number of terms considered is 10: what is the common difference?

Ans. 2.

166. The last principle affords a solution to the following question:—

To find a number m *of arithmetical means between two given numbers* a *and* b.

To resolve this question, it is first necessary to find the common difference. Now we may regard a as the first term of an arithmetical progression, b as a subsequent term, and the required means as intermediate terms. The number of terms of this progression which are considered, will be expressed by $m + 2$.

Now, by substituting in the above formula, b for l, and $m + 2$ for n, it becomes

$$d = \frac{b - a}{m + 2 - 1}, \quad \text{or} \quad d = \frac{b - a}{m + 1};$$

that is, *the common difference* of the required progression is obtained by dividing the difference between the given numbers a and b, by one more than the required number of means.

Having obtained the common difference, form the second term of the progression, or the *first arithmetical mean*, by adding d, or $\frac{b - a}{m + 1}$, to the first term a. The *second mean* is obtained by augmenting the first by d, &c.

1. Find 3 arithmetical means between 2 and 18. The formula

$$d = \frac{b - a}{m + 1}, \quad \text{gives} \quad d = \frac{18 - 2}{4} = 4;$$

hence the progression is

$$2 . 6 . 10 . 14 . 18.$$

2. Find 12 arithmetical means between 12 and 77. The formula

$$d = \frac{b-a}{m+1}, \text{ gives } d = \frac{77-12}{13} = 5;$$

hence the progression is

$$12 . 17 . 22 . 27 . . . . . 72 . 77.$$

167. Remark.—If the same number of arithmetical means are inserted between the terms of a progression, taken two and two, these terms, and the arithmetical means united, will form one and the same progression.

For, let $a . b . c . e . f . . . .$ be the proposed progression, and m the number of means to be inserted between a and b, b and c, c and e. . . . .

From what has just been said, the common difference of each partial progression will be expressed by

$$\frac{b-a}{m+1}, \quad \frac{c-b}{m+1}, \quad \frac{e-c}{m+1} \; . \; . \; . \; .$$

which are equal to each other, since a, b, c, . . . are in progression: therefore, the common difference is the same in each of the partial progressions; and since the *last term* of the first, forms the *first term* of the second, &c., we may conclude that all of these partial progressions form a single progression.

EXAMPLES.

1. Find the sum of the first fifty terms of the progression

$$2 . 9 . 16 . 23 . . .$$

For the 50th term, we have

$$l = 2 + 49 \times 7 = 345.$$

Hence, $$S = (2 + 345) \times \frac{50}{2} = 347 \times 25 = 8675.$$

2. Find the 100th term of the series 2 . 9 . 16 . 23 . . .

Ans. 695.

3. Find the sum of 100 terms of the series 1 . 3 . 5 . 7 . 9 . . .

Ans. 10000.

4. The greatest term considered is 70, the common difference 3, and the number of terms 21: what is the least term and the sum of the series?

Ans. Least term 10; sum of series 840.

5. The first term of a decreasing arithmetical progression is 10, the common difference one third, and the number of terms 21: required the sum of the series. *Ans.* 140.

6. In a progression by differences, having given the common difference 6, the last term 185, and the sum of the terms 2945: find the first term, and the number of terms.

Ans. First term $= 5$; number of terms 31.

7. Find 9 arithmetical means between each antecedent and consequent of the progression $2 \,.\, 5 \,.\, 8 \,.\, 11 \,.\, 14 \ldots$

Ans. $d = 0.3$.

8. Find the number of men contained in a triangular battalion, the first rank containing 1 man, the second 2, the third 3, and so on to the n^{th}, which contains n. In other words, find the expression for the sum of the natural numbers 1, 2, 3, . . . from 1 to n, inclusively. *Ans.* $S = \frac{n(n+1)}{2}$.

9. Find the sum of the n first terms of the progression of uneven numbers 1, 3, 5, 7, 9 . . . *Ans.* $S = n^2$.

10. One hundred stones being placed on the ground, in a straight line, at the distance of 2 yards from each other, how far will a person travel, who shall bring them one by one to a basket, placed at two yards from the first stone?

Ans. 11 miles, 840 yards.

Geometrical Proportion.

168. *Ratio* is the quotient arising from dividing one quantity by another quantity of the same kind. Thus, if A and B represent quantities of the same kind, the ratio of A to B is expressed by

$$\frac{B}{A}.$$

169. If there be four magnitudes, A, B, C, and D, having such values that

$$\frac{B}{A} = \frac{D}{C},$$

then A is said to have the same ratio to B, that C has to D; or, the ratio of A to B is equal to the ratio of C to D. When

four quantities have this relation to each other, they are said to be in proportion. Hence, *proportion is an equality of ratios.*

To express that the ratio of A to B is equal to the ratio of C to D, we write the quantities thus,

$$A : B : : C : D,$$

and read, A is to B, as C is to D.

The quantities which are compared together are called the *terms* of the proportion. The first and last terms are called the *two extremes*, and the second and third terms, the *two means.*

170. Of four proportional quantities, the first and third are called the *antecedents*, and the second and fourth the *consequents;* and the last is said to be a fourth proportional to the other three taken in order.

171. Three quantities are in proportion when the first has the same ratio to the second that the second has to the third; and then the middle term is said to be a mean proportional between the other two.

172. Quantities are said to be in proportion by *inversion*, or *inversely*, when the consequents are made the antecedents and the antecedents the consequents.

173. Quantities are said to be in proportion by *alternation*, or *alternately*, when antecedent is compared with antecedent and consequent with consequent.

174. Quantities are said to be in proportion by *composition*, when the sum of the antecedent and consequent is compared either with antecedent or consequent.

175. Quantities are said to be in proportion by *division*, when the difference of the antecedent and consequent is compared either with antecedent or consequent.

176. Equi-multiples of two or more quantities are the products which arise from multiplying the quantities by the same number. Thus, $m \times A$ and $m \times B$, are equi-multiples of A and B, the common multiplier being m.

177. Two quantities, A and B, are said to be *reciprocally proportional*, or *inversely proportional*, when one increases in the same ratio as the other diminishes. When this relation exists, either of them is equal to a constant quantity divided by the other.

178. If we have the proportion

$$A : B :: C : D,$$

we have $$\frac{B}{A} = \frac{D}{C}, \text{ (Art. 169)};$$

and by clearing the equation of fractions, we have

$$BC = AD; \text{ that is,}$$

Of four proportional quantities, the product of the two extremes is equal to the product of the two means.

179. If four quantities, A, B, C, and D, are so related to each other that

$$A \times D = B \times C,$$

we shall also have, $$\frac{B}{A} = \frac{D}{C},$$

and hence, $A : B :: C : D$; that is

If the product of two quantities is equal to the product of two other quantities, two of them may be made the extremes, and the other two the means of a proportion.

180. If we have three proportional quantities,

$$A : B :: B : C,$$

we have $$\frac{B}{A} = \frac{C}{B};$$

hence, $B^2 = AC$; that is,

The square of the middle term is equal to the product of the two extremes.

181. If we have

$$A : B :: C : D, \text{ and consequently, } \frac{B}{A} = \frac{D}{C},$$

multiplying both members of the equation by $\frac{C}{B}$, we obtain

$$\frac{C}{A} = \frac{D}{B},$$

and hence, $A : C :: B : D$; that is,

If four quantities are proportional, they will be in proportion by alternation.

182. If we have

$$A : B :: C : D, \text{ and } A : B :: E : F,$$

we shall also have

$$\frac{B}{A} = \frac{D}{C} \text{ and } \frac{B}{A} = \frac{F}{E};$$

hence, $\frac{D}{C} = \frac{F}{E}$ and $C : D :: E : F$; that is,

If there are two sets of proportions having an antecedent and consequent in the one equal to an antecedent and consequent of the other, the remaining terms will be proportional.

183. If we have

$$A : B :: C : D, \text{ and consequently, } \frac{B}{A} = \frac{D}{C},$$

we have, by dividing 1 by each member of the equation,

$\frac{A}{B} = \frac{C}{D}$, and consequently, $B : A :: D : C$; that is,

Four proportional quantities will be in proportion, when taken inversely (Art. 172).

184. The proportion

$$A : B :: C : D, \text{ gives } A \times D = B \times C.$$

To each member of the last equation add $B \times D$. We shall then have

$$(A + B) \times D = (C + D) \times B;$$

and by separating the factors, we obtain

$$A + B : B :: C + D : D.$$

If, instead of adding, we subtract $B \times D$ from both members, we have

$$(A - B) \times D = (C - D) \times B;$$

which gives $A - B : B :: C - D : D$; that is,

If four quantities are proportional, they will be in proportion by composition or division.

185. If we have

$$\frac{B}{A} = \frac{D}{C},$$

and multiply the numerator and denominator of the first member by any number m, we obtain

$$\frac{mB}{mA} = \frac{D}{C} \quad \text{and} \quad mA : mB :: C : D; \text{ that is,}$$

Equal multiples of two quantities have the same ratio as the quantities themselves.

186. The proportions

$$A : B :: C : D, \quad \text{and} \quad A : B :: E : F.$$

give $\quad A \times D = B \times C, \quad$ and $\quad A \times F = B \times E;$

adding and subtracting these equations, we obtain

$$A(D \pm F) = B(C \pm E), \text{ or } A : B :: C \pm E : D \pm F; \text{ that is,}$$

If C *and* D, *the antecedent and consequent, be augmented or diminished by quantities* E *and* F, *which have the same ratio as* C *to* D, *the resulting quantities will also have the same ratio.*

187. If we have several proportions,

$$\begin{array}{llll} A : B :: C : D, & \text{which gives} & & A \times D = B \times C, \\ A : B :: E : F, & \text{"} & \text{"} & A \times F = B \times E, \\ A : B :: G : H, & \text{"} & \text{"} & A \times H = B \times G. \\ \text{\&c., \&c.,} & & & \end{array}$$

we shall have, by addition,

$$A(D + F + H) = B(C + E + G);$$

and by separating the factors,

$$A : B : C + E + G : D + F + H; \text{ that is,}$$

In any number of proportions having the same ratio, any antecedent will be to its consequent, as the sum of the antecedents to the sum of the consequents.

188. If we have four proportional quantities

$$A : B :: C : D, \text{ we have } \frac{B}{A} = \frac{D}{C};$$

and raising both members to any power, as the nth, we have

$$\frac{B^n}{A^n} = \frac{D^n}{C^n};$$

and consequently, $\quad A^n : B^n :: C^n : D^n; \;$ that is,

If four quantities are proportional, any like powers or roots will be proportional.

189. Let there be two sets of proportions,

$$A : B : : C : D, \text{ which gives } \frac{B}{A} = \frac{D}{C},$$

$$E : F : : G : H, \quad \text{"} \quad \text{"} \quad \frac{F}{E} = \frac{H}{G}.$$

Multiply them together, member by member, we have

$\dfrac{BF}{AE} = \dfrac{DH}{CG}$, which gives $AE : BF : : CG : DH$; that is,

In two sets of proportional quantities, the products of the corresponding terms will be proportional.

Of Geometrical Progression.

190. In the proportions which have been considered, it has only been required that the ratio of the first term to the second should be the same as that of the third to the fourth. If we impose the farther condition, that the ratio of the second to the third shall also be the same as that of the first to the second, or of the third to the fourth, we shall have a series of numbers, each of which, divided by the preceding one, will give the same ratio. Hence, if any term be multiplied by this quotient, the product will be the succeeding term. A series of numbers so formed is called a *geometrical progression*. Hence,

A *geometrical progréssion*, or *progression by quotients*, is a series of terms, each of which is equal to the product of that which precedes it by a *constant number*, which number is called the *ratio* of the progression. Thus, in the two series,

$$3, \ 6, \ 12, \ 24, \ 48, \ 96, \ldots$$

$$64, \ 16, \ 4, \ 1, \ \frac{1}{4}, \ \frac{1}{16}, \ldots$$

each term of the first contains that which precedes it *twice*, or is equal to double that which precedes it; and each term of the second contains the term which precedes it one-fourth times, or is a *fourth* of that which precedes it. These are geometrical pro-

gressions. In the first, the ratio is 2; in the second, it is $\frac{1}{4}$. The first is called an *increasing* progression, the second a *decreasing* progression.

Let a, b, c, d, e, f, . . . denote numbers in a progression by quotients: they are written thus:

$$a : b : c : d : e : f : g \; . \; . \; .$$

and it is enunciated in the same manner as a progression by differences. It is necessary, however, to make the distinction, that one is a series of *equal differences*, and the other a series of *equal quotients or ratios*. It should be remarked, that each term of the progression is at the same time an antecedent and a consequent, except the first, which is only an antecedent, and the last, which is only a consequent.

191. Let r denote the ratio of the progression

$$a : b : c : d \; . \; . \; .;$$

r being > 1 when the progression is *increasing*, and $r < 1$ when it is *decreasing*. We deduce from the definition, the following equations:

$$b = ar, \quad c = br = ar^2, \quad d = cr = ar^3, \quad e = dr = ar^4 \; . \; . \; .;$$

and, in general, any term n, that is, one which has $n - 1$ terms before it, is expressed by ar^{n-1}.

Let l be this term; we have the formula

$$l = ar^{n-1},$$

by means of which we can obtain any term without being obliged to find all the terms which precede it. That is,

Any term of a geometrical progression is equal to the first term multiplied by the ratio raised to a power whose exponent denotes the number of preceding terms.

EXAMPLES.

1. Find the 5th term of the progression

$$2 : 4 : 8 : 16, \;\&c.,$$

in which the first term is 2, and the common ratio 2.

$$\text{5th term } = 2 \times 2^4 = 2 \times 16 = 32.$$

2. Find the 8th term of the progression

$$2 : 6 : 18 : 54 \ldots$$

$$8\text{th term} = 2 \times 3^7 = 2 \times 2187 = 4374.$$

3. Find the 12th term of the progression

$$64 : 16 : 4 : 1 : \frac{1}{4} \ldots$$

$$12\text{th term} = 64 \left(\frac{1}{4}\right)^{11} = \frac{4^3}{4^{11}} = \frac{1}{4^8} = \frac{1}{65536}.$$

192. We will now explain the method of determining the sum of n terms of the progression

$$a : b : c : d : e : f : \ldots : i : k : l,$$

of which the ratio is r.

If we denote the sum of the series by S, and the nth term by l, we shall have

$$S = a + ar + ar^2 \ldots + ar^{n-2} + ar^{n-1}.$$

If we multiply both members by r, we have

$$Sr = ar + ar^2 + ar^3 \ldots + ar^{n-1} + ar^n;$$

and by subtracting the first equation,

$$Sr - S = ar^n - a, \quad \text{whence,} \quad S = \frac{ar^n - a}{r - 1};$$

and by substituting for ar^n, its value lr, we have

$$S = \frac{lr - a}{r - 1}.$$

That is, to obtain the sum of any number of terms of a progression by quotients, *multiply the last term by the ratio, subtract the first term from this product, and divide the remainder by the ratio diminished by unity.*

EXAMPLES.

1. Find the sum of eight terms of the progression

$$2 : 6 : 18 : 54 : 162 \ldots : 2 \times 3^7 = 4374.$$

$$S = \frac{lr - a}{r - 1} = \frac{13122 - 2}{2} = 6560.$$

2. Find the sum of five terms of the progression

$$2 : 4 : 8 : 16 : 32 ; \ldots .$$

$$S = \frac{lr - a}{r - 1} = \frac{64 - 2}{1} = 62.$$

3. Find the sum of ten terms of the progression

$$2 : 6 : 18 : 54 : 162 \ldots 2 \times 3^9 = 39366.$$

Ans. 59048.

4. What debt may be discharged in a year, or twelve months, by paying \$1 the first month, \$2 the second month, \$4 the third month, and so on, each succeeding payment being double the last; and what will be the last payment?

Ans. Debt, \$4095; last payment, \$2048.

5. A gentleman married his daughter on New-Year's day, and gave her husband 1*s*. toward her portion, and was to double it on the first day of every month during the year: what was her portion? *Ans*. £204 15*s*.

6. A man bought 10 bushels of wheat on the condition that he should pay 1 cent for the first bushel, 3 for the second, 9 for the third, and so on to the last: what did he pay for the last bushel and for the ten bushels?

Ans. Last bushel, \$196,83; total cost, \$295,24.

193. When the progression is decreasing, we have $r < 1$ and $l < a$; the above formula for the sum is then written under the form

$$S = \frac{a - lr}{1 - r},$$

in order that the two terms of the fraction may be positive.

By substituting ar^{n-1} for l in the expression for S, it becomes

$$S = \frac{ar^n - a}{r - 1}, \quad \text{or} \quad S = \frac{a - ar^n}{1 - r}.$$

EXAMPLES.

1. Find the sum of the first five terms of the progression

$$32 : 16 : 8 : 4 : 2.$$

$$S = \frac{a - lr}{1 - r} = \frac{32 - 2 \times \frac{1}{2}}{\frac{1}{2}} = \frac{31}{\frac{1}{2}} = 62.$$

2. Find the sum of the first twelve terms of the progression

$$64 : 16 : 4 : 1 : \frac{1}{4} : \ldots : 64\left(\frac{1}{4}\right)^{11}, \quad \text{or} \quad \frac{1}{65536}.$$

$$S = \frac{a - lr}{1 - r} = \frac{64 - \frac{1}{65536} \times \frac{1}{4}}{\frac{3}{4}} = \frac{256 - \frac{1}{65536}}{3} = 85 + \frac{65535}{196608}.$$

We perceive that the principal difficulty consists in obtaining the numerical value of the last term, a tedious operation, even when the number of terms is not very great.

194. REMARK.—If, in the formula

$$S = \frac{a(r^n - 1)}{r - 1},$$

we suppose $r = 1$, it becomes

$$S = \frac{0}{0}.$$

This result is a symbol of indetermination. It often arises from the existence of a common factor (Art. 113), which becomes nothing by making a particular hypothesis on the quantities which enter the equation. If this common factor can be divided out, the expression will assume a determinate form. This, in fact, is the case in the present question; for, the expression $r^n - 1$ is divisible by $r - 1$ (Art. 61), and gives the quotient

$$r^{n-1} + r^{n-2} + r^{n-3} + \ldots + r + 1;$$

hence, the value of S takes the form

$$S = ar^{n-1} + ar^{n-2} + ar^{n-3} + \ldots + ar + a.$$

Now, making $r = 1$, we have

$$S = a + a + a + \ldots + a = na.$$

We can obtain the same result by going back to the proposed progression

$$a : b : c : \ldots : l,$$

which, in the particular case of $r = 1$, reduces to

$$a : a : a : \ldots : a,$$

the sum of which series is equal to na.

The result $\frac{0}{0}$, given by the formula, may be regarded as indicating that the series is characterized by some particular property. In fact, the progression, being entirely composed of equal terms, is no more a progression by quotients than it is a progression by differences. Therefore, in seeking for the sum of a certain number of the terms, there is no reason for using the formula

$$S = \frac{a(r^n - 1)}{r - 1},$$

in preference to the formula

$$S = \frac{(a + l)n}{2}.$$

which gives the sum in the progression by differences.

195. The consideration of the five quantities, a, r, n, l, and S, which enter into the formulas

$$l = ar^{n-1} \quad \text{and} \quad S = \frac{lr - a}{r - 1},$$

give rise to several curious problems.

Of these cases, we shall consider here, only the most important. We will first find the values of S and r in terms of a, l, and n.

The first formula gives

$$r^{n-1} = \frac{l}{a}, \quad \text{whence} \quad r = \sqrt[n-1]{\frac{l}{a}}.$$

Substituting this value in the second formula, the value of S will be obtained.

The expression

$$r = \sqrt[n-1]{\frac{l}{a}},$$

furnishes the means for resolving the following question, viz.:

To find m *mean proportionals between two given numbers* a *and* b; that is, *to find a number* m *of means, which will form with* a *and* b, *considered as extremes, a progression by quotients.*

To find this series, it is only necessary to know the *ratio*. Now, the required number of means being m, the total number

of terms considered, will be equal to $m+2$. Moreover, we have $l=b$, therefore the value of r becomes

$$r=\sqrt[m+1]{\frac{b}{a}};$$

that is, we must *divide one of the given numbers* (b) *by the other* (a), *then extract that root of the quotient whose index is one more than the required number of means.*

Hence, the progression is

$$a : a\sqrt[m+1]{\frac{b}{a}} : a\sqrt[m+1]{\frac{b^2}{a^2}} : a\sqrt[m+1]{\frac{b^3}{a^3}} : \ldots b.$$

Thus, to insert six mean proportionals between the numbers 3 and 384, we make $m=6$, whence

$$r=\sqrt[7]{\frac{384}{3}}=\sqrt[7]{128}=2;$$

whence we deduce the progression

$$3 : 6 : 12 : 24 : 48 : 96 : 192 : 384.$$

REMARK.—When the same number of mean proportionals are inserted between all the terms of a progression by quotients, taken two and two, all the progressions thus formed will constitute a single progression.

Of Progressions having an infinite Number of Terms.

196. Let there be the decreasing progression

$$a : b : c : d : e : f : \ldots,$$

containing an indefinite number of terms. The formula

$$S=\frac{a-ar^n}{1-r},$$

which represents the sum of n terms, can be put under the form

$$S=\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

Now, since the progression is decreasing, r is a proper fraction, and r^n is also a fraction, which diminishes as n increases. Therefore, the greater the number of terms we take, the more will $\frac{a}{1-r}\times r^n$ diminish, and consequently, the more will the

partial sum of these terms approximate to an equality with the first part of S; that is, to $\frac{a}{1-r}$. Finally, when n is taken greater than any given number, or

$$n = \infty, \quad \text{then} \quad \frac{a}{1-r} \times r^n$$

will be less than any given number, or will become equal to 0; and the expression $\frac{a}{1-r}$ will represent the true value of the sum of all the terms of the series.

Whence, we may conclude, that the expression for *the sum of the terms of a decreasing progression, in which the number of terms is infinite, is*

$$S = \frac{a}{1-r}.$$

This is, properly speaking, the *limit* to which the *partial sums* approach, by taking a greater number of terms of the progression. The number of terms may be taken so great as to make the difference between the sum, and $\frac{a}{1-r}$, as small as we please, and the difference will only become *nothing* when the number of terms taken is infinite.

EXAMPLES.

1. Find the sum of

$$1 : \frac{1}{3} : \frac{1}{9} : \frac{1}{27} : \frac{1}{81} \text{ to infinity.}$$

We have, for the sum of the terms,

$$S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}.$$

2. Again, take the progression

$$1 : \frac{1}{2} : \frac{1}{4} : \frac{1}{8} : \frac{1}{16} : \frac{1}{32} : \&c. \ldots$$

We have $$S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2.$$

What is the error, in each example, for $n = 4$, $n = 5$, $n = 6$?

CHAPTER VIII.

FORMATION OF POWERS, AND EXTRACTION OF ROOTS OF ANY DEGREE. —CALCULUS OF RADICALS.—INDETERMINATE CO-EFFICIENTS.

197. The resolution of equations of the second degree supposes the process for extracting the square root to be known. In like manner, the resolution of equations of the third, fourth, &c. degree, requires that we should know how to extract the third, fourth, &c. root of any numerical or algebraic quantity.

The power of a number can be obtained by the rules of multiplication. and this power is subjected to a certain *law of formation*, which it is necessary to know, in order *to deduce the root from the power.*

Now, the law of formation of the square of a numerical or algebraic quantity, is deduced from the expression for the square of a binomial (Art. 116); so likewise, the law of a power of any degree, is deduced from the same power of a binomial. We shall therefore first determine *the development of any power of a binomial.*

198. By multiplying the binomial $x+a$ into itself several times, the following results are obtained:

$$(x+a) = x + a,$$
$$(x+a)^2 = x^2 + 2ax + a^2,$$
$$(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3,$$
$$(x+a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4,$$
$$(x+a)^5 = x^5 + 5ax^4 + 10a^2x^3 + 10a^3x^2 + 5a^4x + a^5.$$

By examining the developments, we readily discover *the law* according to which the exponents of x decrease and those of a increase, in the successive terms; it is not, however, so easy to

discover a law for the co-efficients. Newton discovered one, by means of which a binomial may be raised to any power, without first obtaining all of the inferior powers. He did not, however, explain the course of reasoning which led him to the discovery; but the law has since been demonstrated in a rigorous manner. Of all the known demonstrations of it, the most elementary is that which is founded upon the *theory of combinations*. However, as the demonstration is rather complicated, we will, in order to simplify it, begin by resolving some problems relative to permutations and combinations, on which the demonstration of the formula for the binomial theorem depends.

Theory of Permutations and Combinations.

199. Let it be proposed to determine the *whole number of ways* in which several letters, a, b, c, d, &c., can be written one after the other. The result corresponding to each change in the position of any one of these letters, is called *a permutation*.

Thus, the two letters a and b furnish the two *permutations*, ab and ba.

In like manner, the three letters, a, b, c, furnish six permutations.

$$\begin{cases} abc \\ acb \\ cab \\ bac \\ bca \\ cba \end{cases}$$

Permutations, *are the results obtained by writing a certain number of letters one after the other, in every possible order, in such a manner that all the letters shall enter into each result, and each letter enter but once.*

PROBLEM 1. *To determine the number of permutations of which* n *letters are susceptible.*

In the first place, two letters, a and b, evidently give two permutations. $\begin{cases} ab \\ ba \end{cases}$

Therefore, the number of permutations of two letters is expressed by 1×2.

Take the three letters, a, b, and c. Reserve either of the letters, as c, and permute the other two, giving $\begin{cases} c \\ ab \\ ba \end{cases}$

Now, the third letter c may be placed before ab, between a and b, and at the right of ab; and the same for ba: that is, in ONE *of the first permutations, the reserved letter c may have three different places, giving three permutations.* Now, as the same may be shown for *each* one of the first permutations, it follows that the whole number of permutations of three letters will be expressed by, $1 \times 2 \times 3$.

$$\left\{\begin{array}{l} cab \\ acb \\ abc \\ cba \\ bca \\ bac \end{array}\right.$$

If now, a fourth letter d be introduced, it can have four places in *each one* of the six permutations of three letters: hence, all the permutations of four letters will be expressed by, $1 \times 2 \times 3 \times 4$.

In general, let there be n letters, a, b, c, &c., and suppose the total number of permutations of $n - 1$ letters to be known; and let Q denote that number. Now, in *each one* of the Q permutations, the reserved letter may have n places, giving n permutations: hence, when it is so combined with all of them, the entire number of permutations will be expressed by $Q \times n$.

Let $n = 2$. Q will then denote the number of permutations that can be made with a single letter; hence, $Q = 1$, and in this particular case we have, $Q \times n = 1 \times 2$.

Let $n = 3$. Q will then express the number of permutations of $3 - 1$ or 2 letters, and is equal to 1×2. Therefore, $Q \times n$ is equal to $1 \times 2 \times 3$.

Let $n = 4$. Q in this case denotes the number of permutations of 3 letters, and is equal to $1 \times 2 \times 3$. Hence, $Q \times n$ becomes $1 \times 2 \times 3 \times 4$; and similarly, when there are more letters.

200. Suppose we have a number m, of letters a, b, c, d, &c. If they are written one after the other, in classes of 2 and 2, or 3 and 3, or 4 and 4 . . . in every possible order in each class, in such a manner, however, that the number of letters in each result shall be less than the number of given letters, we may demand the *whole number* of results thus obtained. These results are called *arrangements.*

Thus, ab, ac, ad, . . . ba, bc, bd, . . . ca, cb, cd, . . . are *arrangements* of m letters taken 2 and 2; or in sets of 2 each.

In like manner, abc, abd, . . . bac, bad, . . . acb, acd, . . . are *arrangements* taken in sets of 3.

Arrangements, *are the results obtained by writing a number* m *of letters one after the other in every possible order, in sets of* 2 *and*

2, 3 *and* 3, 4 *and* 4 . . . n *and* n; m *being* $>$ n; that is, the number of letters in each set being less than the whole number of letters considered. If, however, we suppose $n = m$, the *arrangements* taken n and n, will become simple *permutations*.

PROBLEM 2. *Having given a number* m *of letters* a, b, c, d . . . *to determine the total number of arrangements that may be formed of them by taking them* n *at a time; m being supposed greater than* n.

Let it be proposed, in the first place, to arrange the three letters, a, b, and c, in sets of two each.

First, arrange the letters in sets of one each, and for each set so formed, there will be two letters reserved: the reserved letters for either arrangement, being those which do not enter.

$$\left\{\begin{array}{l} a \\ b \\ c \end{array}\right.$$

When we arrange with reference to a, the reserved letters will be b and c; if with reference to b, the reserved letters will be a and c, &c.

Now, to any one of the letters, as a, annex, in succession, the reserved letters b and c: to the second arrangement b, annex the reserved letters a and c; and to the third arrangement, c, annex the reserved letters a and b: this gives

$$\left\{\begin{array}{l} ab \\ ac \\ ba \\ bc \\ ca \\ cb \end{array}\right.$$

And since each of the first arrangements is repeated as many times as there are reserved letters, it follows, *that the arrangements of three letters taken two in a set, will be equal to the arrangements of the same number of letters taken one in a set, multiplied by the number of reserved letters.*

Let it be required to form the arrangement of four letters, a, b, c, and d, taken 3 in a set.

First, arrange the four letters in sets of two: there will then be two reserved letters. Take *one* of the sets and write after it, in succession, each of the reserved letters: we shall thus form as many sets of three letters each as there are reserved letters; and these sets differ from each other by at least the last letter. Take another of the first arrangements, and annex in succession the reserved letters; we shall again form as many different arrangements as there are reserved letters. Do the same for all of the first arrangements, and it is plain, that the whole number of arrangements

$$\text{In sets of two.}\left\{\begin{array}{l} ab \\ ba \\ ac \\ ca \\ ad \\ da \\ bc \\ cb \\ bd \\ db \\ cd \\ dc \end{array}\right.$$

which will be formed, of four letters, taken 3 and 3, *will be equal to the arrangements of the same letters, taken two in a set, multiplied by the number of reserved letters.*

In order to resolve this question in a general manner, suppose the total number of *arrangements* of m letters, taken $n-1$ in a set, to be known, and denote this number by P.

Take any one of these arrangements, and annex to it, in succession, each of the reserved letters, and of which the number is $m-(n-1)$, or $m-n+1$: it is evident, that we shall thus form a number $m-n+1$ of new arrangements of n letters, each differing from the other by the last letter. Now, take another of the first arrangements of $n-1$ letters, and annex to it, in succession, each of the $m-n+1$ letters which do not make a part of it; we again obtain a number $m-n+1$ of arrangements of n letters, differing from each other, and from those obtained as above, at least in one of the $n-1$ first letters. Now, as we may in the same manner, take all the P arrangements of the m letters, taken $n-1$ in a set, and annex to each in succession each of the $m-n+1$ other letters, it follows that the total number of arrangements of m letters taken n in a set, is expressed by

$$P(m-n+1).$$

To apply this in determining the number of arrangements of m letters, taken 2 and 2, 3 and 3, 4 and 4, or 5 and 5 in a set, make $n=2$; whence, $m-n+1=m-1$; P in this case, will express the total number of arrangements, taken $2-1$ and $2-1$, or 1 and 1; and is consequently equal to m; therefore, the formula becomes $m(m-1)$.

Let $n=3$; whence, $m-n+1=m-2$; P will then express the number of arrangements taken 2 and 2, and is equal to $m(m-1)$; therefore, the formula becomes

$$m(m-1)(m-2).$$

Again, take $n=4$: whence, $m-n+1=m-3$: P will express the number of arrangements taken 3 and 3, or is equal to

$$m(m-1)(m-2);$$

therefore, the formula becomes

$$m(m-1)(m-2)(m-3).$$

REMARK.—From the manner in which these results have been deduced, we conclude that the general formula for m letters taken n in a set, is

$$m(m-1)(m-2)(m-3)\ .\ .\ .\ .\ (m-n+1);$$

that is, *it is composed of the product of the* n *consecutive numbers comprised between* m *and* m − n + 1, *inclusively.*

From this formula, that of the preceding Art. can easily be deduced, viz., the development of the value of $Q \times n$.

For, we see that the *arrangements* become permutations when the number of letters entering into each arrangement is equal to the total number of letters considered.

Therefore, to pass from the total number of arrangements of m letters, taken n and n, to the number of permutations of n letters, it is only necessary to make $m = n$ in the above development, which gives

$$n(n-1)(n-2)(n-3)\ .\ .\ .\ .\ .\ .\ 1.$$

By reversing the order of the factors, and observing that the last is 1, the next to the last 2, the third from the last 3, &c., we have

$$1 \times 2 \times 3 \times 4\ .\ .\ .\ .\ .\ .\ (n-2)(n-1)n,$$

for the development of $Q \times n$.

This is nothing more than the series of natural numbers comprised between 1 and n, inclusively.

201. When the letters are disposed, as in the arrangements, 2 and 2, 3 and 3, 4 and 4, &c., it may be required that no two of the results, thus formed, shall be composed entirely of the same letters, in which case the products of the letters will be different; and we may then demand the whole number of results thus obtained. In this case, the results are called *combinations.*

Thus, *ab*, *ac*, *bc*, . . . *ad*, *bd*, . . . are *combinations* of the letters *a*, *b*, and *c*, &c., taken 2 and 2.

In like manner, *abc*, *abd*, . . . *acd*, *bcd*, . . . are *combinations* of the letters taken 3 and 3.

Combinations, *are arrangements in which any two will differ from each other by at least one of the letters which enter them.*

Hence, there is an essential difference in the signification of the words, *permutations*, *arrangements*, and *combinations.*

PROBLEM 3. *To determine the total number of different combinations that can be formed of* m *letters, taken* n *in a set.*

Let X denote the total number of arrangements that can be formed of m letters, taken n and n: Y the number of permutations of n letters, and Z the total number of *different combinations* taken n and n.

It is evident, that all the possible arrangements of m letters, taken n in a set, can be obtained, by subjecting the n letters of each of the Z combinations, to all the *permutations* of which these letters are susceptible. Now, a single combination of n letters gives, by hypothesis, Y permutations; therefore Z combinations will give $Y \times Z$ arrangements, taken n and n; and as X denotes the total number of arrangements, it follows that the three quantities, X, Y, and Z, give the relations

$$X = Y \times Z; \quad \text{whence,} \quad Z = \frac{X}{Y}.$$

But we have (Art. 200),

$$X = P(m - n + 1),$$

and (Art. 199), $\quad Y = Q \times n;$

therefore, $\quad Z = \dfrac{P(m - n + 1)}{Q \times n} = \dfrac{P}{Q} \times \dfrac{m - n + 1}{n}.$

Since P expresses the total number of arrangements, taken $n - 1$ and $n - 1$, and Q the number of permutations of $n - 1$ letters, it follows that $\frac{P}{Q}$ expresses the number of different combinations of m letters taken $n - 1$ and $n - 1$.

To apply this to the case of the combinations of m letters taken 2 and 2, 3 and 3, 4 and 4, &c.

Make $n = 2$, in which case, $\frac{P}{Q}$ expresses the number of combinations of m letters taken $2 - 1$ and $2 - 1$, or taken 1 and 1, and this number must be equal to m; the above formula therefore becomes

$$m \times \frac{m - 1}{2} \quad \text{or} \quad \frac{m(m - 1)}{1.2}.$$

Let $n = 3$; $\frac{P}{Q}$ will express the number of combinations taken

2 and 2, and is equal to $\frac{m(m-1)}{1.2}$; and the formula becomes

$$\frac{m(m-1)(m-2)}{1.2.3}.$$

In like manner, we find the number of combinations of m letters taken 4 and 4, to be

$$\frac{m(m-1)(m-2)(m-3)}{1.2.3.4};$$

and, in general, the number of combinations of m letters taken n and n, is expressed by

$$\frac{m(m-1)(m-2)(m-3)\ .\ .\ .\ (m-n+1)}{1.2.3.4\ .\ .\ .\ (n-1).n}.$$

which is the development of the expression

$$\frac{P(m-n+1)}{Q \times n}.$$

We may here observe that, if we have a series of numbers, decreasing by unity, and of which the first is m and the last $m-p$, m and p being entire numbers, that the product of these numbers will be exactly divisible by the continued product of all the natural numbers from 1 to $p+1$ inclusively; that is,

$$\frac{m(m-1)(m-2)(m-3)\ .\ .\ .\ (m-p)}{1\ .\ 2\ .\ 3\ .\ 4\ .\ .\ .\ .\ (p+1)}$$

is a whole number. For, from what has been proved, this expression represents the number of different combinations that can be formed of m letters taken in sets of $p+1$ and $p+1$. Now this number of combinations is, from its nature, an entire number; therefore the above expression is necessarily a whole number.

Demonstration of the Binomial Theorem.

202. In order to discover more easily the law for the development of the mth power of the binomial $x+a$, let us observe the law of the product of several binomial factors, $x+a$, $x+b$, $x+c$, $x+d$ . . . of which the first term is the same in each, and the second terms different.

$$
\begin{array}{l}
x + a \\
x + b
\end{array}
$$

1st product - $\begin{array}{l|l} x^2 + a & x + ab \\ + b & \end{array}$

$$
x + c
$$

2d - - - $\begin{array}{l|l|l} x^3 + a & x^2 + ab & x + abc \\ + b & + ac & \\ + c & + bc & \end{array}$

$$
x + d
$$

3d - - - $\begin{array}{l|l|l|l} x^4 + a & x^3 + ab & x^2 + abc & x + abcd \\ + b & + ac & + abd & \\ + c & + ad & + acd & \\ + d & + bc & + bcd & \\ & + bd & & \\ & + cd & & \end{array}$

From these products, obtained by the common rule for algebraic multiplication, we discover the following laws :—

1st. With respect to the exponents, we observe that, the exponent of x, in the first term, is equal to the number of binomial factors employed. In each of the following terms to the right, this exponent diminishes by unity to the last term, where it is 0.

2d. With respect to the co-efficients of the different powers of x, that of the first term is unity; the co-efficient of the second term is equal to the sum of the second terms of the binomials; the co-efficient of the third term is equal to the sum of the products of the different second terms, taken two and two; the co-efficient of the fourth term is equal to the sum of their different products, taken three and three. Reasoning from *analogy*, we may conclude that the co-efficient of the term which has n terms before it, is equal to the sum of the different products of the second terms of the m binomials, taken n and n. The last term of the product is equal to the continued product of the second terms of the binomials.

In order to prove that this law of formation is general, suppose that it has been proved true for a number m of binomials; let us

see if it will continue to be true when the product is multiplied by a new factor.

For this purpose, suppose

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} \ldots + Mx^{m-n+1} + Nx^{m-n} + \ldots + U,$$

to be the product of m binomial-factors, Nx^{m-n} representing the term which has n terms before it, and Mx^{m-n+1} the term which immediately precedes.

Let $x + k$ be the new factor by which we multiply; the product when arranged according to the powers of x, will be

$$\begin{array}{r|r|r|r|r} x^{m+1} + A & x^m + B & x^{m-1} + C & x^{m-2} + \ldots + N & x^{m-n+1} + \ldots \\ + k & + Ak & + Bk & + Mk & + Uk. \end{array}$$

From which we perceive that the *law of the exponents* is evidently the same.

With respect to the co-efficients, we observe,

1st. That the co-efficient of the first term *is unity;* and

2d. That $A + k$, or the co-efficient of x^m, is the *sum of the second terms of the* m + 1 *binomials.*

3d. Since, by hypothesis, B is the sum of the different products of the second terms of the m binomials, taken two and two, and since $A \times k$ expresses the sum of the products of each of the second terms of the m binomials by the new second term k; therefore, $B + Ak$ *is the sum of the different products of the second terms of the* m + 1 *binomials, taken two and two.*

In general, since N expresses the sum of the products of the second terms of the m binomials, taken n and n, and M the sum of their products, taken $n - 1$ and $n - 1$; if we multiply the last set by the new *second term* k, then $N + Mk$, or the co-efficient of the term which has n terms before it, will be equal to the sum of the different products of the second terms of the $m + 1$ binomials, taken n and n. The last term is equal to the continued product of the second terms of the $m + 1$ binomials.

Therefore, the law of composition, supposed true for a number m of binomial factors, is also true for a number denoted by $m + 1$. Hence, it is true for $m + 2$, &c., and is therefore general.

203. Let us now suppose, that in the product resulting from the multiplication of the m binomial factors,

$$x + a, \quad x + b, \quad x + c, \quad x + d, \ldots .$$

we make, $a = b = c = d: \ldots\ldots$

we shall then have

$$(x + a)(x + b)(x + c) \ldots\ldots\ldots = (x + a)^m.$$

The co-efficient of the first term, x^m, will become 1. The co-efficient of x^{m-1}, being $a + b + c + d, \ldots$ will be a taken m times; that is, ma. The co-efficient of x^{m-2}, being

$$ab + ac + ad \ldots\ldots \text{ reduces to } a^2 + a^2 + a^2 \ldots$$

that is, it becomes a^2 taken as many times as there are combinations of m letters, taken two and two, and hence reduces (Art. 201), to

$$m \cdot \frac{m-1}{2} a^2.$$

The co-efficient of x^{m-3} reduces to the product of a^3, multiplied by the number of different combinations of m letters, taken three and three; that is, to

$$m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} a^3, \text{ \&c.}$$

In general, let us denote the term, which has n terms before it, by Nx^{m-n}. Then, the co-efficient N will denote the sum of the products of the second terms, taken n and n; and when all of the terms are supposed equal, it becomes equal to a^n multiplied by the number of different combinations that can be made with m letters, taken n and n. Therefore, the co-efficient of the general term (Art. 201), is

$$N = \frac{P(m-n+1)}{Q \times n} a^n,$$

from which we deduce the formula,

$$(x + a)^m = x^m + max^{m-1} + m \cdot \frac{m-1}{2} a^2x^{m-2}$$

$$+ m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} a^3x^{m-3} \ldots + \frac{P(m-n+1)}{Q \cdot n} a^nx^{m-n} \ldots + a^m.$$

The term

$$\frac{P(m-n+1)}{Qn} a^nx^{m-n}$$

is called *the general term*, because by making $n = 2, 3, 4, \ldots$ all of the others can be deduced from it. The term which immediately precedes it, is evidently,

$$\frac{P}{Q} a^{n-1}x^{m-n+1}, \text{ since } \frac{P}{Q}$$

expresses the number of combinations of m letters taken $n-1$ and $n-1$. Hence, we see, that the co-efficient

$$\frac{P(m-n+1)}{Q\times n},$$

is equal to the co-efficient $\frac{P}{Q}$ of the preceding term, multiplied by $m-n+1$, the exponent of x in that term, and divided by n, the number of terms preceding the required term.

Since $\frac{P}{Q}$ is the co-efficient of the preceding term, we may, by observing how the co-efficients are formed from each other, express the co-efficient of the general term thus,

$$N=\frac{m(m-1)(m-2)(m-3)\ .\ .\ .\ (m-n+2)(m-n+1)}{1\ .\ 2\ .\ 3\ .\ 4\ .\ .\ (n-1)\ .\ n}.$$

The *simple law*, demonstrated above, enables us to determine the co-efficient of any term from the co-efficient of the preceding term.

The co-efficient of any term is formed by multiplying the co-efficient of the preceding term by the exponent of x *in that term, and dividing the product by the number of terms which precede the required term.*

For example, let it be required to develop $(x+a)^6$.

From this law, we have,

$$(x+a)^6=x^6+6ax^5+15a^2x^4+20a^3x^3+15a^4x^2+6a^5x+a^6.$$

After having formed the first two terms from the general formula $x^m+max^{m-1}+,\ .\ .\ .$ multiply 6, the co-efficient of the second term, by 5, the exponent of x in that term, and then divide the product by 2, which gives 15 for the co-efficient of the third term. To obtain that of the fourth, multiply 15 by 4, the exponent of x in the third term, and divide the product by 3, the number of terms which precede the fourth; this gives 20; and the co-efficients of the other terms are found in the same way.

In like manner, we find

$$(x+a)^{10}=x^{10}+10ax^9+45a^2x^8+120a^3x^7+210a^4x^6$$
$$+252a^5x^5+210a^6x^4+120a^7x^3+45a^8x^2+10a^9x+a^{10}.$$

204. It frequently occurs that the terms of the binomial are affected with co-efficients and exponents, as in the following example:

Let it be required to raise the binomial

$$3a^2c - 2bd$$

to the fourth power.

Placing $3a^2c = x$ and $-2bd = y$, we have

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4;$$

and substituting for x and y their values, we have

$$(3a^2c - 2bd)^4 = (3a^2c)^4 + 4(3a^2c)^3(-2bd) + 6(3a^2c)^2(-2bd)^2 + 4(3a^2c)(-2bd)^3 + (-2bd)^4,$$

or, by performing the operations indicated,

$$(3a^2c - 2bd)^4 = 81a^8c^4 - 216a^6c^3bd + 216a^4c^2b^2d^2 - 96a^2cb^3d^3 + 16b^4d^4.$$

The terms of the development are alternately plus and minus, as they should be, since the second term is —.

205. The powers of any polynomial, may easily be found by the binomial theorem. For example, raise

$$a + b + c$$

to the third power.

First, put $b + c = d$.

Then $(a+b+c)^3 = (a+d)^3 = a^3 + 3a^2d + 3ad^2 + d^3;$

and by substituting for the value of d,

$$\begin{aligned}(a+b+c)^3 = a^3 + 3a^2b &+ 3ab^2 + b^3 \\ 3a^2c &+ 3b^2c + 6abc \\ &+ 3ac^2 + 3bc^2 \\ &+ c^3.\end{aligned}$$

This development is composed of *the cubes of the three terms, plus three times the square of each term by the first powers of the two others, plus six times the product of all three terms.*

To apply the preceding formula to the development of the cube of a trinomial, in which the terms are affected with co-efficients and exponents, *designate each term by a single letter, and perform the operations indicated; then replace the letters introduced by their values.*

From this rule, we will find that

$$(2a^2 - 4ab + 3b^2)^3 = 8a^6 - 48a^5b + 132a^4b^2 - 208a^3b^3 + 198a^2b^4 - 108ab^5 + 27b^6.$$

The fourth, fifth, &c. powers of any polynomial can be developed in a similar manner.

Consequences of the Binomial Formula.

206. The development of the binomial expression $(x + a)^m$ will always contain $m + 1$ terms. Hence, if we take that term of the development which has n terms before it, the number of terms after it will be expressed by $m - n$.

Let us now seek the co-efficient of the term which has n terms after it, and which, consequently, has $m - n$ terms before it. We obtain this co-efficient by simply substituting $m - n$ for n, in the last value of N in Art. 203. We then have,

$$N = \frac{m(m-1)(m-2)\,.\,.\,(n+2)(n+1)}{1\,.\,2\,.\,3\,.\,.\,.\,(m-n-1)(m-n)}.$$

As we can always take the term which has n terms before it, nearer to the first term than the one which has $m - n$ terms before it, we will examine that part of the co-efficient which is derived from the terms lying between these two. We may write

$$N = \frac{m(m-1)\ldots(m-n+1)\,.\,(m-n)\,.\,(m-n-1)\ldots(n+2)\,.\,(n+1)}{1\,.\,2 \qquad n \qquad (n+1)\,.\,(n+2)\,.\,.\,(m-n-1)\,.\,(m-n)}.$$

Now, by cancelling the like factors in the numerator and denominator, we have

$$N = \frac{m(m-1)\,.\,.\,.\,.\,(m-n+1)}{1\,.\,2\,.\,.\,.\,.\,.\,.\,n}: \text{ hence,}$$

In the development of any power of a binomial, the co-efficients at equal distances from the two extremes are equal to each other.

207. If we designate by K the co-efficient of the term which has n terms before it, that term will be expressed by Ka^nx^{m-n}; and the corresponding term counted from the last term of the series, will be $Ka^{m-n}x^n$.

Now, the first co-efficient expresses the number of different combinations that can be formed with m letters taken n and n; and the second, the number which can be formed when taken $m - n$

and $m - n$; we may therefore conclude that, *the number of different combinations of* m *letters taken* n *and* n, *is equal to the number of combinations of* m *letters taken* m − n *and* m − n.

For example, *twelve* letters combined 5 and 5, give the same number of combinations as when taken 12 − 5 and 12 − 5, or 7 and 7. Five letters combined 2 and 2, give the same number of combinations as when combined 5 − 2 and 5 − 2, or 3 and 3.

208. If, in the general formula,

$$(x + a)^m = x^m + max^{m-1} + m\frac{m-1}{2}a^2x^{m-2} +, \text{ \&c.},$$

we suppose $x = 1$, $a = 1$, we have,

$$(1 + 1)^m \text{ or } 2^m = 1 + m + m\frac{m-1}{2} + m\frac{m-1}{2}\cdot\frac{m-2}{3} +, \text{ \&c.}$$

That is, *the sum of the co-efficients of all the terms of the formula for the binomial, is equal to the* m*th power of* 2.

Thus, in the particular case

$$(x + a)^5 = x^5 + 5ax^4 + 10a^2x^3 + 10a^3x^2 + 5a^4x + a^5,$$

the sum of the co-efficients

$$1 + 5 + 10 + 10 + 5 + 1$$

is equal to $2^5 = 32$. In the 10th power developed, the sum of the co-efficients is equal to $2^{10} = 1024$.

Extraction of the Cube Root of Numbers.

209. The *cube* or *third power* of a number, is the product which arises from multiplying the number twice by itself. The *cube root*, or *third root* of a number is either of three equal factors into which it may be resolved; and hence, to extract the cube root, is to seek one of these factors.

Every number which can be resolved into three equal factors that are commensurable with unity, is called a *perfect cube;* and any number which cannot be so resolved, is called an *imperfect cube.*

The first ten numbers are

roots,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10;
cubes,	1,	8,	27,	64,	125,	216,	343,	512,	729,	1000.

Reciprocally, the numbers of the first line are the cube roots of the numbers of the second

We perceive, by inspection, that there are but nine *perfect cubes* among all the numbers expressed by one, two, and three figures. Every other number, except the nine written above, which can be expressed by one, two, or three figures, will be an imperfect cube; and hence, its cube root will be expressed by a whole number, plus an irrational number, as may be shown by a course of reasoning entirely similar to that pursued in the latter part of Art. 118.

210. Let us find the difference between the cubes of two consecutive numbers.

Let a and $a+1$, be two consecutive whole numbers; we have

$$(a+1)^3 = a^3 + 3a^2 + 3a + 1;$$

whence, $$(a+1)^3 - a^3 = 3a^2 + 3a + 1.$$

That is, *the difference between the cubes of two consecutive whole numbers, is equal to three times the square of the least number, plus three times the number, plus* 1.

Thus, the difference between the cube of 90 and the cube of 89, is equal to

$$3(89)^2 + 3 \times 89 + 1 = 24031.$$

211. In order to extract the cube root of an entire number, we will observe, that when the figures expressing the number do not exceed three, the entire part of the root is found by comparing the number with the first nine perfect cubes. For example, the cube root of 27 is 3. The cube root of 30 is 3, plus an irrational number, less than unity. The cube root of 72 is 4, plus an irrational number less than unity, since 72 lies between the perfect cubes 64 and 125.

When the number is expressed by more than three figures, the process will be as follows. Let the proposed number be 103823.

$$\begin{array}{rr|l} & 10\dot{3}\,82\dot{3} & 47 \\ & 64 & \overline{8} \\ \cline{2-2} 4^2 \times 3 = 48 & 398.23 & \end{array} \qquad \begin{array}{r} 48 \\ 48 \\ \hline 384 \\ 192 \\ \hline 2304 \\ 48 \\ \hline 18432 \\ 9216 \\ \hline 110592 \end{array} \qquad \begin{array}{r} 47 \\ 47 \\ \hline 329 \\ 188 \\ \hline 2209 \\ 47 \\ \hline 15463 \\ 8836 \\ \hline 103823 \end{array}$$

This number being comprised between 1,000, which is the cube

of 10, and 1,000,000, which is the cube of 100, its root will be expressed by two figures, or by tens and units. Denoting the tens by a, and the units by b, we have (Art. 198),

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Whence it follows, that the cube of a number composed of tens and units, is made up of four distinct parts: viz., *the cube of the tens, three times the product of the square of the tens by the units, three times the product of the tens by the square of the units, and the cube of the units.*

Now, the cube of the tens, giving at least, *thousands*, the last three figures to the right cannot form a part of it: the cube of tens must therefore be found in the part 103 which is separated from the last three figures. The root of the greatest cube contained in 103 being 4, this is the number of tens in the required root. Indeed, 103823 is evidently comprised between $(40)^3$ or 64,000, and $(50)^3$ or 125,000; hence, the required root is composed of 4 tens, plus a certain number of units less than *ten*.

Having found the number of tens, subtract its cube, 64, from 103, and there remains 39, to which bring down the part 823, and we have 39823, which contains *three times the square of the tens by the units*, plus the two parts named above. Now, as the square of tens gives at least hundreds, it follows that three times the square of the tens by the units, must be found in the part 398, to the left of 23, which is separated from it by a point. Therefore, dividing 398 by 48, which is three times the square of the tens, the quotient 8 will be the units of the root, or something greater, since 398 hundreds is composed of three times the square of the tens by the units, together with the two other parts.

We may ascertain whether the figure 8 is too great, by forming from the 4 tens and 8 units the three parts which enter into 39823; but it is much easier to cube 48, as has been done in the above table. Now, the cube of 48 is 110592, which is greater than 103823; therefore, 8 is too great. By cubing 47 we obtain 103823; hence the proposed number is a perfect cube, and 47 is its cube root.

REMARK I.—The units figures could not be first obtained, because the cube of the units might give tens, and even hundreds, and the tens and hundreds would be confounded with those which arise from other parts of the cube.

REMARK II.—The operations in the last example have been performed on but two periods. It is plain, however, that the same reasoning is equally applicable to larger numbers; for, by changing the order of the units, we do not change the relation in which they stand to each other.

Thus, in the number 43 725 658, the two periods 43 725, have the same relation to each other, as in the number 43725; and hence, the methods pursued in the last example are equally applicable to larger numbers.

212. Hence, for the extraction of the cube root of numbers, we have the following

RULE.

I. *Separate the given number into periods of three figures each beginning at the right hand: the left-hand period will often contain less than three places of figures.*

II. *Seek the greatest cube in the first period, at the left, and set its root on the right, after the manner of a quotient in division. Subtract the cube of this figure of the root from the first period, and to the remainder bring down the first figure of the next period, and call this number the dividend.*

III. *Take three times the square of the root just found for a divisor, and see how often it is contained in the dividend, and place the quotient for a second figure of the root. Then cube the figures of the root thus found, and if their cube be greater than the first two periods of the given number, diminish the last figure; but if it be less, subtract it from the first two periods, and to the remainder bring down the first figure of the next period, for a new dividend.*

IV. *Take three times the square of the whole root for a new divisor, and seek how often it is contained in the new dividend; the quotient will be the third figure of the root. Cube the whole root, and subtract the result from the first three periods of the given number, and proceed in a similar way for all the periods.*

REMARK.—If any of the remainders are equal to, or exceed, *three times the square of the root obtained plus three times this root, plus one*, the last figure of the root is too small and must be augmented by at least unity (Art. 210).

EXAMPLES.

1. $\sqrt[3]{48228544} = 364.$
2. $\sqrt[3]{27054036008} = 3002.$
3. $\sqrt[3]{483249} = 78$, with a remainder 8697.
4. $\sqrt[3]{91632508641} = 4508$, with a remainder 20644129.
5. $\sqrt[3]{32977340218432} = 32068.$

To extract the n^{th} Root of a whole Number.

213. The n^{th} root of a number, is one of the n equal factors into which the number may be resolved. If the factors are commensurable with unity, the number is said to be a *perfect power*, if they are not commensurable with unity, the number is said to be an *imperfect power*.

In order to generalize the process for *the extraction of roots*, we will denote the proposed number by N, and the degree of the root to be extracted by n. If the number of figures in N, does not exceed n, the rational part of the root will be expressed by a single figure.

Having formed the n^{th} power of all the numbers from 1 to 9, inclusive, compare the given number with these powers, and the root of the one next less, will be that part of the required root which can be expressed by a whole number; for, the n^{th} power of 9 is the largest number which can be expressed by n figures.

When N contains more than n figures, there will be more than one figure in the root, which may then be considered as composed of tens and units. Designating the tens by a, and the units by b, we have (Art. 203),

$$N = (a+b)^n = a^n + na^{n-1}b + n\frac{n-1}{2}a^{n-2}b^2 +, \text{ \&c.};$$

that is, the proposed number contains *the* n[th] *power of the tens, plus* n *times the product of the* n —1[th] *power of the tens by the units*, plus a series of other parts which it is not necessary to consider.

Now, as the n^{th} power of the tens, cannot give units of an order inferior to 1 followed by n ciphers, the last n figures on the right, cannot make a part of it. They must then be pointed off

and the root of the greatest n^{th} power contained in the figures on the left should be extracted: this root will be *the tens of the required root.*

If this part on the left should contain more than n figures, the n figures on the right of it, must be separated from the rest, and the root of the greatest n^{th} power contained in the part on the left extracted, and so on. Hence the following

RULE.

I. *Divide the number* N *into periods of* n *figures each, beginning at the right hand; extract the root of the greatest* n^{th} *power contained in the left-hand period, and subtract the* n^{th} *power of this figure from the left-hand period.*

II. *Bring down to the right of the remainder derived from the left-hand period, the first figure of the next period, and call this number the dividend.*

III. *Form the* $n-1$ *power of the first figure of the root, multiply it by* n, *and see how often the product is contained in the dividend: the quotient will be the second figure of the root, or something greater.*

IV. *Raise the number thus formed to the* n^{th} *power, then subtract this result from the two left-hand periods, and to the new remainder bring down the first figure of the next period: then divide the number thus formed by* n *times the* $n-1$ *power of the two figures of the root already found, and continue this operation until all the periods are brought down.*

EXAMPLES.

1. What is the fourth root of 531441?

$$
\begin{array}{rl}
 & \dot{5}3\,144\dot{1} \mid 27 \\
2^4 = & 16 \\
4 \times 2^3 = 32 \mid & \overline{371} \\
(27)^4 = & 531441.
\end{array}
$$

We first divide off, from the right hand, the period of four figures, and then find the greatest fourth root contained in 53, the first period to the left, which is 2. We next subtract the 4th power of 2, which is 16, from 53, and to the remainder 37 we bring down the first figure of the next period. We then divide 371 by

4 times the cube of 2, which gives 11 for a quotient: but this we know is too large. By trying the numbers 9 and 8, we find them also too large: then trying 7, we find the exact root to be 27.

214. REMARK.—When the degree of the root to be extracted is a multiple of two or more numbers, as 4, 6, ..., *the root can be obtained by extracting the roots of more simple degrees, successively.* To explain this, we will remark that,

$$(a^3)^4 = a^3 \times a^3 \times a^3 \times a^3 = a^{3+3+3+3} = a^{3\times 4} = a^{12}.$$

and that in general (Art. 13),

$$(a^m)^n = a^m \times a^m \times a^m \times a^m \; . \; . \; . = a^{m\times n}:$$

hence, *the* nth *power of the* m th *power of a number, is equal to the* mn th *power of this number.*

Let us see if the reciprocal of this is also true.

Let $$\sqrt[n]{\sqrt[m]{a}} = a';$$

then raising both members to the n^{th} power, we have, from the definition of the n^{th} root,

$$\sqrt[m]{a} = a'^n;$$

and by raising both members of the last equation to the m^{th} power

$$a = (a'^n)^m = a'^{mn}.$$

Extracting the mn^{th} root of the last equation, we have

$$\sqrt[mn]{a} = a';$$

and hence, $$\sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a},$$

since each is equal to a'. Therefore, *the* n th *root of the* m th *root of any number, is equal to the* mn th *root of that number.* And in a similar manner, it might be proved that

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}.$$

By this method we find that

1. $\sqrt[4]{256} = \sqrt{\sqrt{256}} = \sqrt{16} = 4.$

2. $\sqrt[6]{2985984} = \sqrt[3]{\sqrt{2985984}} = \sqrt[3]{1728} = 12.$

3. $\sqrt[6]{1771561} = \sqrt[3]{\sqrt{1771561}} = 11.$

4. $\sqrt[8]{1679616} = \sqrt[4]{1296} = \sqrt{\sqrt{1296}} = 6.$

REMARK.—Although the successive roots may be extracted in any order whatever, it is better to extract the roots of the lowest degree first, for then the extraction of the roots of the higher degrees, which is a more complicated operation, is effected upon numbers containing fewer figures than the proposed number.

Extraction of Roots by Approximation.

215. When it is required to extract the n^{th} root of a number which is not a *perfect power*, the method already explained, will give only the entire part of the root, or the root to within unity. As to the number which is to be added, in order to complete the root, it cannot be obtained exactly, but we can approximate to it as near as we please.

Let it be required to extract the n^{th} root of the whole number a to within a fraction $\frac{1}{p}$; that is, so near, that the error shall be less than $\frac{1}{p}$.

We will observe that we can write

$$a = \frac{ap^n}{p^n}.$$

If we denote by r, the root of ap^n to within unity, the number $\frac{a \times p^n}{p^n} = a$, will be comprehended between $\frac{r^n}{p^n}$ and $\frac{(r+1)^n}{p^n}$; therefore the $\sqrt[n]{a}$ will be comprised between the two numbers, $\frac{r}{p}$ and $\frac{r+1}{p}$; and consequently, their difference $\frac{1}{p}$ will be greater than the difference between $\frac{r}{p}$ and the true root. Hence, $\frac{r}{p}$ is the required root to within the fraction $\frac{1}{p}$.

Hence, *to extract the* nth *root of a whole number to within a fraction* $\frac{1}{p}$, *multiply the number by* p^n; *extract the* nth *root of the product to within unity, and divide the result by* p.

216. Again, suppose it is required to extract the n^{th} root of the fraction $\frac{a}{b}$.

Multiply each term of the fraction by

b^{n-1}, and it becomes $\frac{a}{b} = \frac{ab^{n-1}}{b^n}$.

Let r denote the n^{th} root of ab^{n-1}, to within unity;

$\frac{ab^{n-1}}{b^n} = \frac{a}{b}$, will be comprised between $\frac{r^n}{b^n}$ and $\frac{(r+1)^n}{b^n}$;

and consequently, $\frac{r}{b}$ will be the n^{th} root of $\frac{a}{b}$, to within the fraction $\frac{1}{b}$.

Therefore, *after having made the denominator of the fraction a perfect power of the* n^th^ *degree, extract the* n^th^ *root of the numerator, to within unity, and divide the result by the root of the new denominator.*

When a greater degree of exactness is required than that indicated by $\frac{1}{b}$, extract the root of ab^{n-1} to within any fraction $\frac{1}{p}$; and designate this root by $\frac{r'}{p}$. Now, since $\frac{r'}{p}$ is the root of the numerator to within $\frac{1}{p}$, it follows, that $\frac{r'}{bp}$ is the true root of the fraction to within $\frac{1}{bp}$.

EXAMPLES.

1. Suppose it were required to extract the cube root of 15, to within $\frac{1}{12}$. We have

$$15 \times 12^3 = 15 \times 1728 = 25920.$$

Now the cube root of 25920, to within unity, is 29; hence the required root is,

$$\frac{29}{12} = 2\frac{5}{12}.$$

2. Extract the cube root of 47, to within $\frac{1}{20}$.

We have

$$47 \times 20^3 = 47 \times 8000 = 376000.$$

Now the cube root of 376000, to within unity, is 72; hence

$$\sqrt[3]{47} = \frac{72}{20} = 3\frac{12}{20}, \text{ to within } \frac{1}{20}.$$

3. Find the value of $\sqrt[3]{25}$ to within 0.001.

To do this, multiply 25 by the cube of 1000, or 1000000000, which gives 25000000000. Now, the cube root of this number, is 2920; hence

$$\sqrt[3]{25} = 2.920 \text{ to within } 0.001.$$

217. REMARK.—In general, *in order to extract the cube root of a whole number to within a given decimal fraction, annex three times as many ciphers to the number, as there are decimal places in the required root; extract the cube root of the number thus formed to within unity, and point off from the right of this root the required number of decimals.*

218. We will now explain the method of *extracting the cube root of a decimal fraction.* Suppose it is required to extract the cube root of 3.1415.

Since the denominator, 10000, of this fraction, is not a perfect cube, make it one, by multiplying it by 100; *this is equivalent to annexing two ciphers to the proposed decimal,* which then becomes, 3.141500. Extract the cube root of 3141500, that is, of the number considered independent of the comma, to within unity; this gives 146. Then divide by 100, or $\sqrt[3]{1000000}$, and we find

$$\sqrt[3]{3.1415} = 1.46 \text{ to within } 0.01.$$

Hence, to extract the cube root of a decimal number, we have the following

RULE.

Annex ciphers to the decimal part, if necessary, until it can be divided into exact periods of three figures each, observing that the number of periods must be made equal to the number of decimal places required in the root. Then, extract the root as in entire numbers, and point off as many places for decimals as there are periods in the decimal part of the number.

To extract the cube root of a vulgar fraction to within a given decimal fraction, the most simple method is to *reduce the proposed fraction to a decimal fraction, continuing the operation until*

the number of decimal places is equal to three times the number required in the root. The question is then reduced to extracting the cube root of a decimal fraction.

219. Suppose it is required to find the sixth root of 23, to within 0.01.

Applying the rule of Art. 215 to this example, we multiply 23 by 100^6, or annex *twelve* ciphers to 23; then extract the sixth root of the number thus formed to within unity, and divide this root by 100, or point off two decimals on the right.

We thus find that $\sqrt[6]{23} = 1.68$, to within 0.01.

EXAMPLES.

1. Find the $\sqrt[3]{473}$ to within $\frac{1}{20}$. *Ans.* $7\frac{3}{4}$.
2. Find the $\sqrt[3]{79}$ to within .0001. *Ans.* 4.2908.
3. Find the $\sqrt[6]{13}$ to within .01. *Ans.* 1.53.
4. Find the $\sqrt[3]{3.00415}$ to within .0001. *Ans.* 1.4429.
5. Find the $\sqrt[3]{0.00101}$ to within .01. *Ans.* 0.10.
6. Find the $\sqrt[3]{\frac{14}{25}}$ to within .001. *Ans.* 0.824.

Extraction of Roots of Algebraic Quantities.

220. Before extracting the root of an algebraic quantity, let us see in what manner any power of it may be formed.

Let it be required to form the fifth power of $2a^3b^2$. We have

$$(2a^3b^2)^5 = 2a^3b^2 \times 2a^3b^2 \times 2a^3b^2 \times 2a^3b^2 \times 2a^3b^2,$$

from which it follows, 1st. That the co-efficient 2 must be multiplied by itself four times, or raised to the 5th power. 2d. That each of the exponents of the letters must be added to itself four times, or multiplied by 5.

Hence, $\qquad (2a^3b^2)^5 = 2^5 \,.\, a^{3\times5}b^{2\times5} = 32a^{15}b^{10}.$

In like manner, $\quad (8a^2b^3c)^3 = 8^3 \,.\, a^{2\times3}b^{3\times3}c^3 = 512a^6b^9c^3.$

Therefore, in order to raise a monomial to any power, *raise the co-efficient to this power, and multiply the exponent of each of the letters by the exponent of the power, and unite the terms.*

Hence, to extract any root of a monomial,

1st. *Extract the root of the co-efficient and divide the exponent of each letter by the index of the root.* 2d. *To the root of the co-*

efficient annex each letter with its new exponent, and the result will be the required root. Thus,

$$\sqrt[3]{64a^9b^3c^6} = 4a^3bc^2; \quad \sqrt[4]{16a^8b^{12}c^4} = 2a^2b^3c.$$

From this rule we perceive, that in order that a monomial may be a perfect power, 1st, its co-efficient must be a perfect power; and 2d, the exponent of each letter must be divisible by the *index* of the root to be extracted. It will be shown hereafter, how the expression for the root of a quantity, which is not a perfect power, is reduced to its simplest terms.

221. Hitherto, in finding the power of a monomial, we have paid no attention to the sign with which the monomial may be affected. It has already been shown, that whatever be the sign of a monomial, *its square is always positive.*

Let n be any whole number; then, every power of an even degree, as $2n$, can be considered as the n^{th} power of the square; that is, $(a^2)^n = a^{2n}$.

Hence, it follows, *that every power of an even degree, will be essentially positive, whether the quantity itself be positive or negative.*

Thus, $$(\pm 2a^2b^3c)^4 = +16a^8b^{12}c^4.$$

Again, as every power of an uneven degree, $2n+1$, is but the product of the power of an even degree, $2n$, by the first power; it follows that, *every power of an uneven degree, of a monomial, is affected with the same sign as the monomial itself.*

Hence, $(+4a^2b)^3 = +64a^6b^3$; and $(-4a^2b)^3 = -64a^6b^3$

From the preceding reasonings, we conclude,

1st. *That when the degree of the root of a monomial is uneven, the root will be affected with the same sign as the monomial.*

Hence,

$$\sqrt[3]{+8a^3} = +2a; \quad \sqrt[3]{-8a^3} = -2a; \quad \sqrt[5]{-32a^{10}b^5} = -2a^2b.$$

2d. *When the degree of the root is even, and the monomial a positive quantity, the root is affected either with the sign + or —.*

Thus, $$\sqrt[4]{81a^4b^{12}} = \pm 3ab^3; \quad \sqrt[6]{64a^{18}} = \pm 2a^3.$$

3d. *When the degree of the root is even, and the monomial negative, the root is impossible;* for, there is no quantity which, being

raised to a power of an even degree, will give a negative result. Therefore,

$$\sqrt[4]{-a}, \quad \sqrt[6]{-b}, \quad \sqrt[8]{-c},$$

are symbols of operation which it is impossible to execute. They are *imaginary expressions* (Art. 126), like

$$\sqrt{-a}, \quad \sqrt{-b},$$

EXAMPLES.

1. What is the cube root of $8a^6b^3c^{12}$? *Ans.* $2a^2bc^4$.
2. What is the 4th root of $81a^4b^8c^{16}$? *Ans.* $3ab^2c^4$.
3. What is the 5th root of $-32a^5c^{10}d^{15}$? *Ans.* $-2ac^2d^3$.
4. What is the cube root of $-125a^9b^6c^3$? *Ans.* $-5a^3b^2c$.

Extraction of Roots of Polynomials.

222. Let us first examine the law of formation of any power of a polynomial. To begin with a simple example, let us develop

$$(a + y + z)^3.$$

If we place $y + z = u$, we shall have,

$$(a + u)^3 = a^3 + 3a^2u + 3au^2 + u^3;$$

or by replacing u by its value, $y + z$,

$$(a + y + z)^3 = a^3 + 3a^2(y + z) + 3a(y + z)^2 + (y + z)^3;$$

or performing the operations indicated,

$$(a + y + z)^3 = a^3 + 3a^2y + 3a^2z + 3ay^2 + 6ayz + 3az^2 + y^3 + 3y^2z + 3yz^2 + z^3.$$

When the polynomial is composed of more than three terms, as $a + y + z + x \ldots . p$, let, as before, $u =$ the sum of all the terms after the first. Then, $a + u$ will be equal to the given polynomial, and

$$(a + u)^3 = a^3 + 3a^2u + 3au^2 + u^3;$$

from which we see, that *the cube of any polynomial is equal to the cube of the first term, plus three times the square of the first term multiplied by each of the remaining terms, plus other terms.*

If u does not contain a, it is plain that the exponent of a in each term, as a^3, $3a^2u$, &c., will be greater than in any of the following terms; and hence, *every term will be irreducible with the terms which precede or follow it.*

If u contains a, as in the polynomial

$$a^2 + ax + b, \quad \text{where} \quad u = ax + b,$$

the terms will still be irreducible with each other, provided we arrange the polynomial with reference to the letter a. For, if the given polynomial be arranged with reference to a, the exponent of a in the first term will be greater than the exponent of a in u: hence, its cube will contain a with a greater exponent than will result from multiplying its square by u. Also, the co-efficient of u multiplied by the first term of u, will contain a to a higher power than any of the following terms of the development, and hence, will be irreducible with them; and the same may be shown for the subsequent terms.

In order to extract any root of a polynomial, we will first explain the method of extracting the cube root. It will then be easy to generalize this method, and apply it to the case of any root whatever.

Let N be any polynomial, and R its cube root. Suppose the two polynomials to be arranged with reference to some letter, as a, for example. It results from the law of formation of the cube of a polynomial (Art. 222), that in the cube of R, the cube of the first term, and three times the square of the first term by the second, cannot be reduced with each other, nor with any of the following terms.

Hence, the cube root of that term of N which contains a, affected with the highest exponent, will be the first term of R; and the second term of R will be found by dividing the second term of N by three times the square of the first term of R.

By examining the development of the trinomial $a + y + z$, we see, that if we form the cube of the two terms of the root found as above, and subtract it from N, and then divide the first term of the remainder by 3 times the square of the first term of R, the quotient will be the third term of the root. Therefore, having arranged the terms of N, with reference to any letter, we have, for the extraction of the cube root, the following

RULE.

I. *Extract the cube root of the first term.*

II. *Divide the second term of* N *by three times the square of the first term of* R; *the quotient will be the second term of* R.

III. *Having found the first two terms of* R, *form the cube of this binomial and subtract it from* N; *after which, divide the first term of the remainder by three times the square of the first term of* R: *the quotient will be the third term of* R.

IV. *Cube the three terms of the root found, and subtract the cube from* N: *then divide the first term of the remainder by the divisor already used, and the quotient will be the fourth term of the root: the remaining terms, if there are any, may be found in a similar manner.*

EXAMPLES.

1. Extract the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

$$\begin{array}{lll} & x^6-6x^5+15x^4-20x^3+15x^2-6x+1 & \underline{|\,x^2-2x+1} \\ (x^2-2x)^3 = & \underline{x^6-6x^5+12x^4-\ \ 8x^3} & \quad 3x^4 \\ \text{1st rem.} & \qquad 3x^4-12x^3+, \&c. & \\ (x^2-2x+1)^3 = & x^6-6x^5+15x^4-20x^3+15x^2-6x+1. & \end{array}$$

In this example, we first extract the cube root of x^6, which gives x^2, for the first term of the root. Squaring x^2, and multiplying by 3, we obtain the divisor $3x^4$: this is contained in the second term $-6x^5$, $-2x$ times. Then cubing the root, and subtracting, we find that the first term of the remainder $3x^4$, contains the divisor once. Cubing the whole root, we find the cube equal to the given polynomial. Hence, x^2-2x+1, is the exact cube root.

2. Find the cube root of

$$x^6+6x^5-40x^3+96x-64.$$

3. Find the cube root of

$$8x^6-12x^5+30x^4-25x^3+30x^2-12x+8$$

223. The rule for the extraction of the cube root is easily extended to a root with a higher index. For,

Let $\quad a+b+c+\ .\ .\ .\ f$, be any polynomial.

Let $s=$ the sum of all the terms after the first.

Then $a+s=$ the given polynomial; and

$$(a+s)^n = a^n + na^{n-1}s + \text{ other terms.}$$

That is, *the* n^th^ *power of a polynomial, is equal to the* n^th^ *power of the first term, plus* n *times the first term raised to the power*

$n-1$, *multiplied by each of the remaining terms,* + *other terms of the development.*

Hence, we see, that the rule for the cube root will become the rule for the n^{th} root, by first extracting the n^{th} root of the first term, taking for a divisor n times this root raised to the $n-1$ power, and raising the partial roots to the n^{th} power, instead of to the cube.

EXAMPLES.

1. Extract the 4th root of

$$16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4.$$

$$\begin{array}{r|l} 16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4 & 2a - 3x \\ \hline (2a-3x)^4 = 16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4 & 4 \times (2a)^3 = 32a^3 \end{array}$$

We first extract the 4th root of $16a^4$, which is $2a$. We then raise $2a$ to the third power, and multiply by 4, the index of the root; this gives the divisor $32a^3$. This divisor is contained in the second term $-96a^3x$, $-3x$ times, which is the second term of the root. Raising the whole root to the 4th power, we find the power equal to the given polynomial.

2. What is the 4th root of the polynomial,

$$81a^8c^4 + 16b^4d^4 - 96a^2cb^3d^3 - 216a^6c^3bd + 216a^4c^2b^2d^2.$$

3. Find the 5th root of

$$32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1.$$

Calculus of Radicals.

224. When the monomial or polynomial whose root is to be extracted, cannot be resolved into as many equal and rational factors as there are units in the index of the root, it is said to be an *imperfect power.* The root is then indicated by placing the quantity under the radical sign, and writing over it at the left hand, the index of the root. Thus, the fourth root of $3ab^2 + 9ac^5$, is written

$$\sqrt[4]{3ab^2 + 9ac^5}.$$

The index of the root is also called the index of the radical. It is plain that a monomial will be a perfect power, *when the numerical co-efficient is a perfect power, and the exponent of each letter exactly divisible by the index of the root.*

By the definition of a root (Art. 213), we have

$$(\sqrt[n]{abc\ldots})^n = abc\ldots;$$

and by the rule for the raising of powers,

$$(\sqrt[n]{a}\times\sqrt[n]{b}\times\sqrt[n]{c}\ldots)^n = (\sqrt[n]{a})^n\times(\sqrt[n]{b})^n\times(\sqrt[n]{c})^n\ldots = abc\ldots;$$

and since the n^{th} powers are equal, the quantities themselves are equal: hence,

$$\sqrt[n]{abc}\ \ldots\ldots = \sqrt[n]{a}\times\sqrt[n]{b}\times\sqrt[n]{c}\ \ldots\ldots$$

that is, *the* n$^{\text{th}}$ *root of the product of any number of factors, is equal to the product of their* n$^{\text{th}}$ *roots.*

1. Let us apply the above principle in reducing to its simplest form the imperfect power, $\sqrt[3]{54a^4b^3c^2}$. We have

$$\sqrt[3]{54a^4b^3c^2} = \sqrt[3]{27a^3b^3}\times\sqrt[3]{2ac^2} = 3ab\sqrt[3]{2ac^2}.$$

2. In like manner,

$$\sqrt[3]{8a^2} = 2\sqrt[3]{a^2};\ \text{ and }\ \sqrt[4]{48a^5b^8c^6} = 2ab^2c\sqrt[4]{3ac^2};$$

3. Also,

$$\sqrt[6]{192a^7bc^{12}} = \sqrt[6]{64a^6c^{12}}\times\sqrt[6]{3ab} = 2ac^2\sqrt[6]{3ab}.$$

In the expressions, $3ab\sqrt[3]{2ac^2}$, $2\sqrt[3]{a^2}$, $2ab^2c\sqrt[4]{3ac^2}$, each quantity placed before the radical, is called a *co-efficient* of the radical.

225. The rule of Art. 214 gives rise to another kind of simplification.

Take, for example, the radical expression, $\sqrt[6]{4a^2}$; from this rule, we have

$$\sqrt[6]{4a^2} = \sqrt[3]{\sqrt{4a^2}},$$

and as the quantity affected with the radical of the second degree, $\sqrt{\ }$, is a perfect square, its root can be extracted: hence,

$$\sqrt[6]{4a^2} = \sqrt[3]{2a}.$$

In like manner,

$$\sqrt[4]{36a^2b^2} = \sqrt{\sqrt{36a^2b^2}} = \sqrt{6ab}.$$

In general,

$$\sqrt[mn]{a^n} = \sqrt[m]{\sqrt[n]{a^n}} = \sqrt[m]{a};$$

that is; when the index of a radical is a multiple of any number n, and the quantity under the radical sign is an exact n^{th} power, *we can, without changing the value of the radical, divide its index by* n, *and extract the* n^{th} *root of the quantity under the sign.*

This proposition is the inverse of another, not less important; viz., *the index of a radical may be multiplied by any number, provided we raise the quantity under the sign to a power of which this number is the exponent.*

For, since a is the same thing as $\sqrt[n]{a^n}$, we have,

$$\sqrt[m]{a} = \sqrt[m]{\sqrt[n]{a^n}} = \sqrt[mn]{a^n}.$$

226. This last principle serves to reduce two or more radicals to a common index.

For example, let it be required to reduce the two radicals

$$\sqrt[3]{2a} \quad \text{and} \quad \sqrt[4]{(a+b)}$$

to the same index.

By multiplying the index of the first by 4, the index of the second, and raising the quantity $2a$ to the fourth power; then multiplying the index of the second by 3, the index of the first, and cubing $a+b$, the value of neither radical will be changed, and the expressions will become

$$\sqrt[3]{2a} = \sqrt[12]{2^4a^4} = \sqrt[12]{16a^4}; \quad \text{and} \quad \sqrt[4]{(a+b)} = \sqrt[12]{(a+b)^3}.$$

Hence, to reduce radicals to a common index, we have the following

RULE.

Multiply the index of each radical by the product of the indices of all the other radicals, and raise the quantity under each radical sign to a power denoted by this product.

This rule, which is analogous to that given for the reduction of fractions to a common denominator, is susceptible of similar modifications.

For example, reduce the radicals

$$\sqrt[4]{a}, \quad \sqrt[6]{5b}, \quad \sqrt[8]{a^2+b^2},$$

to the same index.

Since 24 is the least common multiple of the indices 4, 6, and 8, it is only necessary to multiply the first by 6, the second by

4, and the third by 3, and to raise the quantities under each radical sign to the 6th, 4th, and 3d powers respectively, which gives

$$\sqrt[4]{a}=\sqrt[24]{a^6}; \quad \sqrt[6]{5b}=\sqrt[24]{5^4b^4}, \quad \sqrt[8]{a^2+b^2}=\sqrt[24]{(a^2+b^2)^3}.$$

In applying the above rules to numerical examples, beginners very often make mistakes similar to the following: viz., in reducing the radicals $\sqrt[3]{2}$ and $\sqrt{3}$ to a common index, after having multiplied the index of the first, by that of the second, and the index of the second by that of the first, then, instead of multiplying the *exponent* of the quantity under the first sign by 2, and the *exponent* of that under the second by 3, they often multiply the *quantity* under the first sign by 2, and the *quantity* under the second by 3. Thus, they would have

$$\sqrt[3]{2}=\sqrt[6]{2\times 2}=\sqrt[6]{4}, \quad \text{and} \quad \sqrt{3}=\sqrt[6]{3\times 3}=\sqrt[6]{9}.$$

Whereas, they should have, by the foregoing rule,

$$\sqrt[3]{2}=\sqrt[6]{(2)^2}=\sqrt[6]{4}, \quad \text{and} \quad \sqrt{3}=\sqrt[6]{(3)^3}=\sqrt[6]{27}.$$

Reduce $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt[5]{\tfrac{1}{2}}$, to the same index.

Addition and Subtraction of Radicals.

227. Two radicals are *similar*, when they have the same index, and the same quantity under the sign. Thus,

$$3\sqrt{ab} \text{ and } 7\sqrt{ab}; \text{ as also, } 3a^2\sqrt[3]{b^2}, \text{ and } 9c^3\sqrt[3]{b^2},$$

are similar radicals.

In order to add or subtract similar radicals, *add or subtract their co-efficients, and to the sum or difference annex the common radical.* Thus,

$$3\sqrt[3]{b}+2\sqrt[3]{b}=5\sqrt[3]{b}; \text{ also, } 3\sqrt[3]{b}-2\sqrt[3]{b}=\sqrt[3]{b}.$$

Again, $\quad 3a\sqrt[4]{b}\pm 2c\sqrt[4]{b}=(3a\pm 2c)\sqrt[4]{b}.$

Dissimilar radicals may sometimes be reduced to similar radicals, by the rules of Arts. 224 and 225. For example,

1 $\quad \sqrt{48ab^2}+b\sqrt{75a}=4b\sqrt{3a}+5b\sqrt{3a}=9b\sqrt{3a}.$

2. $\sqrt[3]{8a^3b + 16a^4} - \sqrt[3]{b^4 + 2ab^3} = 2a\sqrt[3]{b + 2a} - b\sqrt[3]{b + 2a};$

$$= (2a - b)\sqrt[3]{b + 2a}.$$

3. $3\sqrt[6]{4a^2} + 2\sqrt[3]{2a} = 3\sqrt[3]{2a} + 2\sqrt[3]{2a} = 5\sqrt[3]{2a}.$

When the radicals are dissimilar and irreducible, they can only be added or subtracted, by means of the signs $+$ or $-$.

Multiplication and Division.

228. We will suppose that the radicals have been reduced to a common index.

Let it be required to multiply $\sqrt[n]{a}$ by $\sqrt[n]{b}$.

If we denote the product by P, we have

$$\sqrt[n]{a} \times \sqrt[n]{b} = P;$$

and by raising both members to the n^{th} power,

$$(\sqrt[n]{a})^n \times (\sqrt[n]{b})^n = ab = P^n;$$

and by extracting the n^{th} root,

$$\sqrt[n]{a} \times \sqrt[n]{b} = P = \sqrt[n]{ab};$$

that is, *the product of the* n^th^ *roots of two quantities, is equal to the* n^th^ *root of their product.*

Let it be required to divide $\sqrt[n]{a}$ by $\sqrt[n]{b}$.

If we designate the quotient by Q, we have

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = Q;$$

and by raising both members to the n^{th} power,

$$\frac{(\sqrt[n]{a})^n}{(\sqrt[n]{b})^n} = \frac{a}{b} = Q^n;$$

and by extracting the n^{th} root,

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = Q = \sqrt[n]{\frac{a}{b}};$$

that is, *the quotient of the* n^th^ *roots of two quantities, is equal to the* n^th^ *root of their quotient.*

Therefore, for the multiplication and division of radicals, we have the following

RULE.

I. *Reduce the radicals to a common index.*

II. *If the radicals have co-efficients, first multiply or divide them separately.*

III. *Multiply or divide the quantities under the radical sign by each other, and prefix to the product or quotient, the common radical sign.*

EXAMPLES.

1. The product of

$$2a\sqrt[3]{\frac{a^2+b^2}{c}} \times -3a\sqrt[3]{\frac{(a^2+b^2)^2}{d}} = -6a^2\sqrt[3]{\frac{(a^2+b^2)^3}{cd}}$$

$$= -\frac{6a^2(a^2+b^2)}{\sqrt[3]{cd}}.$$

2. The product of

$$3a\sqrt[4]{8a^2} \times 2b\sqrt[4]{4a^2c} = 6ab\sqrt[4]{32a^4c} = 12a^2b\sqrt[4]{2c}.$$

3. The quotient of

$$\frac{\sqrt[3]{a^2b^2+b^4}}{\sqrt[3]{\dfrac{a^2-b^2}{8b}}} = \sqrt[3]{\frac{8b(a^2b^2+b^4)}{a^2-b^2}} = 2b\sqrt[3]{\frac{a^2+b^2}{a^2-b^2}}.$$

4. The product of

$$3a\sqrt[6]{b} \times 5b\sqrt[8]{2c} = 15ab \times \sqrt[24]{8b^4c^3}.$$

5. Multiply $\sqrt{2} \times \sqrt[3]{3}$ by $\sqrt[4]{\frac{1}{2}} \times \sqrt[3]{\frac{1}{3}}$.

Ans. $\sqrt[12]{8}$.

6. Multiply $2\sqrt{15}$ by $3\sqrt[3]{10}$.

Ans. $6\sqrt[6]{337500}$.

7. Multiply $4\sqrt[5]{\frac{2}{3}}$ by $2\sqrt{\frac{3}{4}}$.

Ans. $8\sqrt[10]{\frac{27}{256}}$.

8. Reduce $\dfrac{2\sqrt{3} \times \sqrt[3]{4}}{\frac{1}{2}\sqrt[4]{2} \times \sqrt[3]{3}}$ to its lowest terms.

Ans. $4\sqrt[12]{288}$

9. Reduce $\sqrt{\dfrac{\sqrt{\frac{1}{2}} \times 2\sqrt[3]{3}}{4\sqrt[3]{2} \times \sqrt{3}}}$ to its lowest terms.

Ans. $\dfrac{1}{2}\sqrt[12]{\dfrac{2}{3}}$

10. Multiply $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$, together.

Ans. $\sqrt[12]{648000}$.

11. Multiply $\sqrt[7]{\dfrac{4}{3}}$, $\sqrt[3]{\dfrac{1}{2}}$, and $\sqrt[14]{6}$, together.

Ans. $\sqrt[42]{\dfrac{2}{27}}$.

12. Multiply $\left(4\sqrt{\dfrac{7}{3}} + 5\sqrt{\dfrac{1}{2}}\right)$ by $\left(\sqrt{\dfrac{7}{3}} + 2\sqrt{\dfrac{1}{2}}\right)$.

Ans. $\dfrac{43}{3} + \dfrac{13}{6}\sqrt{42}$.

13. Divide $\dfrac{1}{2}\sqrt{\dfrac{1}{2}}$ by $\left(\sqrt{2} + 3\sqrt{\dfrac{1}{2}}\right)$

Ans. $\dfrac{1}{10}$.

14. Divide 1 by $\sqrt[4]{a} + \sqrt[4]{b}$.

Ans. $\dfrac{\sqrt[4]{a^3} - \sqrt[4]{a^2b} + \sqrt[4]{ab^2} - \sqrt[4]{b^3}}{a - b}$.

15. Divide $\sqrt[4]{a} + \sqrt[4]{b}$ by $\sqrt[4]{a} - \sqrt[4]{b}$.

Ans. $\dfrac{a + b + 2\sqrt{ab} + 2\sqrt[4]{a^3b} + 2\sqrt[4]{ab^3}}{a - b}$.

Powers and Roots of Radicals.

229. By raising $\sqrt[m]{a}$ to the n^{th} power, we have

$$(\sqrt[m]{a})^n = \sqrt[m]{a} \times \sqrt[m]{a} \times \sqrt[m]{a} \ldots = \sqrt[m]{a^n},$$

by the rule just given for the multiplication of radicals. Hence, for raising a radical to any power, we have the following

RULE.

Raise the quantity under the sign to the given power, and affect the result with the radical sign, having the primitive index. If it has a co-efficient, first raise it to the given power.

EXAMPLES.

1. $(\sqrt[4]{4a^3})^2 = \sqrt[4]{(4a^3)^2} = \sqrt[4]{16a^6} = 2a\sqrt[4]{a^2}.$

2. $(3\sqrt[3]{2a})^5 = 3^5 . \sqrt[3]{(2a)^5} = 243\sqrt[3]{32a^5} = 486a\sqrt[3]{4a^2}.$

When the index of the radical is a multiple of the power to which it is to be raised, the result can be simplified.

For, $\sqrt[4]{2a} = \sqrt{\sqrt{2a}}$ (Art. 214): hence, in order to square $\sqrt[4]{2a}$, we have only to omit the first radical, which gives

$$(\sqrt[4]{2a})^2 = \sqrt{2a}.$$

Again, to square $\sqrt[6]{3b}$, we have $\sqrt[6]{3b} = \sqrt{\sqrt[3]{3b}}$: hence,

$$(\sqrt[6]{3b})^2 = \sqrt[3]{3b}.$$

Consequently, *when the index of the radical is divisible by the exponent of the power to which it is to be raised, perform the division, leaving the quantity under the radical sign unchanged.*

Let it be required to extract the m^{th} root of the radical $\sqrt[n]{a}$. We have (Art. 214),

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}.$$

Hence, to extract the root of a radical, *multiply the index of the radical by the index of the root to be extracted, leaving the quantity under the sign unchanged.*

This rule is nothing more than the principle of Art. 214, enunciated in an inverse order.

1. $\sqrt[3]{\sqrt[4]{3c}} = \sqrt[12]{3c}$; and $\sqrt{\sqrt[3]{5c}} = \sqrt[6]{5c}.$

When the quantity under the radical is a perfect power, of the *degree of either of the roots to be extracted*, the result can be simplified.

Thus, $\sqrt[3]{\sqrt[4]{8a^3}} = \sqrt[4]{\sqrt[3]{8a^3}} = \sqrt[4]{2a}.$

In like manner, $\sqrt{\sqrt[5]{9a^2}} = \sqrt[5]{\sqrt{9a^2}} = \sqrt[5]{3a}.$

230. The rules just demonstrated for the calculus of radicals, depend upon the fact, that the n^{th} root of the product of several factors, is equal to the product of the n^{th} roots of these factors. This, however, has been proved on the supposition that, *when the powers of the same degree of two expressions are equal, the expressions themselves are also equal.* Now, this last proposition, which is true for absolute numbers, is not always true for algebraic expressions; for it is easily shown that the same number can have *more than one square root, cube root, fourth root, &c.*

Let us denote the algebraic value of the square root of a by x, and the *arithmetical* value of it by p; we have the equations

$$x^2 = a, \quad \text{and} \quad x^2 = p^2, \quad \text{whence} \quad x = \pm p.$$

Hence we see, that the square of p, (which is the root of a), will give a, whether its sign be $+$ or $-$.

In the second place, let x be the algebraic value of the cube root of a, and p the numerical value of this root; we have the equations

$$x^3 = a, \quad \text{and} \quad x^3 = p^3.$$

The last equation is satisfied by making $x = p$.

Observing that the equation $x^3 = p^3$ can be put under the form $x^3 - p^3 = 0$, and that the expression $x^3 - p^3$ is divisible by $x - p$ (Art. 61), which gives the exact quotient, $x^2 + px + p^2$, the above equation can be transformed into

$$(x - p)\,(x^2 + px + p^2) = 0.$$

Now, every value of x which will satisfy this equation, will satisfy the first equation. But this equation can be satisfied by supposing

$$x - p = 0, \quad \text{whence} \quad x = p;$$

or by supposing

$$x^2 + px + p^2 = 0,$$

from which last, we have

$$x = -\frac{p}{2} \pm \frac{p}{2}\sqrt{-3}, \quad \text{or} \quad x = p\left(\frac{-1 \pm \sqrt{-3}}{2}\right).$$

Hence, *the cube root of* a, *admits of three different algebraic values, viz.*,

$$p, \quad p\left(\frac{-1+\sqrt{-3}}{2}\right), \quad \text{and} \quad p\left(\frac{-1-\sqrt{-3}}{2}\right).$$

Again, resolve the equation

$$x^4 = p^4,$$

in which p denotes the arithmetical value of $\sqrt[4]{a}$. This equation can be put under the form

$$x^4 - p^4 = 0;$$

which reduces to

$$(x^2 - p^2)(x^2 + p^2) = 0;$$

and this equation can be satisfied, by supposing

$$x^2 - p^2 = 0, \quad \text{whence} \quad x = \pm p;$$

or by supposing

$$x^2 + p^2 = 0, \quad \text{whence} \quad x = \pm\sqrt{-p^2} = \pm p\sqrt{-1}.$$

We therefore obtain four *different algebraic* expressions for the fourth root of a.

As another example, resolve the equation

$$x^6 - p^6 = 0.$$

This equation can be put under the form

$$(x^3 - p^3)(x^3 + p^3) = 0;$$

which may be satisfied by making either of the factors equal to zero.

But $\qquad x^3 - p^3 = 0$, gives

$$x = p, \quad \text{and} \quad x = p\left(\frac{-1 \pm \sqrt{-3}}{2}\right).$$

And if in the equation $x^3 + p^3 = 0$, we make $p = -p'$, it becomes $x^3 - p'^3 = 0$, from which we deduce

$$x = p', \quad \text{and} \quad x = p'\left(\frac{-1 \pm \sqrt{-3}}{2}\right);$$

or, substituting for p' its value $-p$,

$$x = -p, \quad \text{and} \quad x = -p\left(\frac{-1 \pm \sqrt{-3}}{2}\right).$$

Therefore, the value of x in the equation $x^6 - p^6 = 0$, and con-

sequently, the 6th root of a, admits of *six* values. If we make

$$\alpha = \frac{-1+\sqrt{-3}}{2}, \quad \text{and} \quad \alpha' = \frac{-1-\sqrt{-3}}{2},$$

these values may be expressed by

$$p, \quad \alpha p, \quad \alpha' p, \quad -p - \alpha p - \alpha' p.$$

We may then conclude from analogy, that in every equation of the form $x^m - a \pm 0$, or $x^m - p^m = 0$, x is susceptible of m different values; that is, the m^{th} root of a number admits of m *different algebraic values.*

231. If, in the preceding equations, and the results corresponding to them, we suppose, as a particular case, $a = 1$, whence $p = 1$, we shall obtain the second, third, fourth, &c. roots of unity. Thus $+1$ and -1 are the two *square roots of unity,* because the equation $x^2 - 1 = 0$, gives $x = \pm 1$.

In like manner,

$$1 +, \quad \frac{-1+\sqrt{-3}}{2}, \quad \frac{-1-\sqrt{-3}}{2},$$

are the three cube roots of unity, or the roots of $x^3 - 1 = 0$; and

$$+1, \quad -1, \quad +\sqrt{-1}, \quad -\sqrt{-1},$$

are the four fourth roots of unity, or the roots of $x^4 - 1 = 0$.

232. It results from the preceding analysis, that the rules for the calculus of radicals, which are exact when applied to absolute numbers, are susceptible of some *modifications,* when applied to *expressions or symbols which are purely algebraic;* these modifieations are more particularly necessary when applied to *imaginary expressions,* and are a consequence of what has been said in Art. 230.

For example, the product of

$$\sqrt{-a} \quad \text{by} \quad \sqrt{-a},$$

by the rule of Art. 228, would be

$$\sqrt{-a} \times \sqrt{-a} = \sqrt{+a^2}.$$

Now, $\sqrt{a^2}$ is equal to $\pm a$ (Art. 139); there is, then, apparently, an uncertainty as to the sign with which a should be affected. Nevertheless, the true answer is $-a$; for, in order

to square $\sqrt{m}$, it is only necessary to suppress the radical; but

$$\sqrt{-a} \times \sqrt{-a} = (\sqrt{-a})^2 = -a.$$

Again, let it be required to form the product

$$\sqrt{-a} \times \sqrt{-b}.$$

By the rule of Art. 228, we shall have

$$\sqrt{-a} \times \sqrt{-b} = \sqrt{+ab}.$$

Now, $\sqrt{ab} = \pm p$ (Art. 230), p being the arithmetical value of the square root of ab; but the true result is $-p$ or $-\sqrt{ab}$, so long as both the radicals $\sqrt{-a}$ and $\sqrt{-b}$ are affected with the sign $+$.

For, $\sqrt{-a} = \sqrt{a} . \sqrt{-1}$; and $\sqrt{-b} = \sqrt{b} . \sqrt{-1}$, hence,

$$\sqrt{-a} \times \sqrt{-b} = \sqrt{a} . \sqrt{-1} \times \sqrt{b} \times \sqrt{-1} = \sqrt{ab}\,(\sqrt{-1})^2$$
$$= \sqrt{ab} \times -1 = -\sqrt{ab}.$$

By similar methods we find the different powers of $\sqrt{-1}$ to be as follows:—

1. $\sqrt{-1} \times \sqrt{-1} = (\sqrt{-1})^2 = -1.$

2. $(\sqrt{-1})^3 = (\sqrt{-1})^2 . \sqrt{-1} = -\sqrt{-1}.$

3. $(\sqrt{-1})^4 = (\sqrt{-1})^2 . (\sqrt{-1})^2 = -1 \times -1 = +1.$

Again, let it be proposed to determine the product of $\sqrt[4]{-a}$ by the $\sqrt[4]{-b}$ which, from the rule, would be $\sqrt[4]{+ab}$, and consequently, would give the four values (Art. 231),

$$+\sqrt[4]{ab}, \quad -\sqrt[4]{ab}, \quad +\sqrt[4]{ab} . \sqrt{-1}, \quad -\sqrt[4]{ab} . \sqrt{-1}.$$

To determine the true product, observe that

$$\sqrt[4]{-a} = \sqrt[4]{a} . \sqrt[4]{-1}, \quad \text{and} \quad \sqrt[4]{-b} = \sqrt[4]{b} . \sqrt[4]{-1}.$$

But, $\sqrt[4]{-1} \times \sqrt[4]{-1} = (\sqrt[4]{-1})^2 = \left(\sqrt{\sqrt{-1}}\right)^2 = \sqrt{-1}$;

hence $\sqrt[4]{-a} . \sqrt[4]{-b} = \sqrt[4]{ab} . \sqrt{-1}.$

We will apply the preceding calculus to the verification of the expression

$$\frac{-1+\sqrt{-3}}{2},$$

considered as a root of the equation $x^3 - 1 = 0$; that is, as one of the cube roots of 1 (Art. 230).

From the formula,

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

we have $$\left(\frac{-1+\sqrt{-3}}{2}\right)^3$$

$$= \frac{(-1)^3 + 3(-1)^2.\sqrt{-3} + 3(-1).(\sqrt{-3})^2 + (\sqrt{-3})^3}{8}$$

$$= \frac{-1 + 3\sqrt{-3} - 3 \times -3 - 3\sqrt{-3}}{8} = \frac{8}{8} = 1.$$

The second value, $\dfrac{-1-\sqrt{-3}}{2}$, may be verified in the same manner. It should be remarked, that either of the imaginary roots is the square of the other; a fact which may be easily verified.

Theory of Exponents.

233. In extracting the n^{th} root of a quantity a^m, we have seen that when m is a multiple of n, we should divide the exponent m by n the index of the root. When m is not divisible by n, the operation of extracting the root is indicated by indicating the division of the two exponents. Thus,

$$\sqrt[n]{a^m} = a^{\frac{m}{n}},$$

a notation founded on the rule for the exponents, in the extraction of the roots of monomials. *In such expressions, the numerator indicates the power to which the quantity is to be raised, and the denominator, the root to be extracted.*

Therefore, $\sqrt[3]{a^2} = a^{\frac{2}{3}}$; and $\sqrt[4]{a^7} = a^{\frac{7}{4}}$.

If it is required to divide a^m by a^n, in which m and n are positive whole numbers, we know that the exponent of the divisor should be subtracted from the exponent of the dividend, and we have

$$\frac{a^m}{a^n} = a^{m-n}.$$

If $m > n$, the division will be exact; but when $m < n$, the division cannot be effected, but still we subtract, in the algebraic sense, the exponent of the divisor from that of the dividend. Let p be the arithmetical difference between n and m; then will

$$n = m + p, \quad \text{whence} \quad \frac{a^m}{a^{m+p}} = a^{-p};$$

but
$$\frac{a^m}{a^{m+p}} = \frac{1}{a^p}; \quad \text{hence,} \quad a^{-p} = \frac{1}{a^p}.$$

Therefore, the *expression* a^{-p} is the symbol of a division which has not been performed; and its true value is the quotient represented by unity, divided by a, affected with the exponent p, taken positively. Thus,

$$a^{-3} = \frac{1}{a^3}; \quad \text{and} \quad a^{-5} = \frac{1}{a^5}.$$

Since, $a^{-p} = \dfrac{1}{a^p}$; and $\dfrac{1}{a^{-p}} = p^n$, we conclude that,

Any factor may be transferred from the numerator to the denominator, or from the denominator to the numerator, by changing the sign of its exponent.

If it is required to extract the n^{th} root of $\dfrac{1}{a^m}$, we have

$$\frac{1}{a^m} = a^{-m}; \quad \text{hence,} \quad \sqrt[n]{\frac{1}{a^m}} = \sqrt[n]{a^{-m}} = a^{\frac{-m}{n}}.$$

The notation of fractional exponents, whether positive or negative, has the advantage of giving an entire form to all expressions whose roots or powers are to be indicated.

From the *conventional expressions founded on the preceding rules,* we have

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}; \quad a^{-p} = \frac{1}{a^p}; \quad \text{and} \quad \sqrt[n]{\frac{1}{a^m}} = a^{-\frac{m}{n}}.$$

We may therefore substitute the second value in each expression, for the first, or reciprocally.

As a^p is called a to the p power, when p is a positive whole number, so, by analogy,

$$a^{\frac{m}{n}}, \quad a^{-p}, \quad a^{-\frac{m}{n}},$$

are called, respectively, a to the $\dfrac{m}{n}$ power, a to the $-p$ power,

a to the $-\frac{m}{n}$ power, in which algebraists have generalized the word *power*. It would, perhaps, be more accurate to say, a, exponent $\frac{m}{n}$, a, exponent $-p$, and a, exponent $-\frac{m}{n}$; using the word *power* only when we wish to designate the product of a number multiplied by itself two or more times.

Multiplication of Quantities affected with any Exponents.

234. In order to multiply $a^{\frac{3}{5}}$ by $a^{\frac{2}{3}}$, it is only necessary *to add the two exponents*, and we have

$$a^{\frac{3}{5}} \times a^{\frac{2}{3}} = a^{\frac{3}{5}+\frac{2}{3}} = a^{\frac{19}{15}}.$$

For, by Art. 233,

$$a^{\frac{3}{5}} = \sqrt[5]{a^3}; \quad \text{and} \quad a^{\frac{2}{3}} = \sqrt[3]{a^2};$$

hence, $$a^{\frac{3}{5}} \times a^{\frac{2}{3}} = \sqrt[5]{a^3} \times \sqrt[3]{a^2};$$

reducing to a common index (Art. 226), and then multiplying,

$$a^{\frac{3}{5}} \times a^{\frac{2}{3}} = \sqrt[15]{a^{19}} = a^{\frac{19}{15}}.$$

Again, multiply $a^{-\frac{3}{4}}$ by $a^{\frac{5}{6}}$.

We have, $a^{-\frac{3}{4}} = \sqrt[4]{\frac{1}{a^3}}$, and $a^{\frac{5}{6}} = \sqrt[6]{a^5}$;

hence,

$$a^{-\frac{3}{4}} \times a^{\frac{5}{6}} = \sqrt[4]{\frac{1}{a^3}} \times \sqrt[6]{a^5} = \sqrt[12]{\frac{1}{a^9}} \times \sqrt[12]{a^{10}} = \sqrt[12]{\frac{a^{10}}{a^9}} = \sqrt[12]{a} = a^{\frac{1}{12}};$$

and consequently,

$$a^{-\frac{3}{4}} \times a^{\frac{5}{6}} = a^{-\frac{3}{4}+\frac{5}{6}} = a^{-\frac{9}{12}+\frac{10}{12}} = a^{\frac{1}{12}}.$$

In general, multiplying $a^{-\frac{m}{n}}$ by $a^{\frac{p}{q}}$; we have

$$a^{-\frac{m}{n}} \times a^{\frac{p}{q}} = a^{-\frac{m}{n}+\frac{p}{q}} = a^{\frac{np-mq}{nq}}.$$

Therefore, in order to multiply two monomials affected with any exponents whatever, *follow the rule given* in Art. 41, for quantities affected with entire exponents.

From this rule, we shall find

1. $a^{\frac{3}{4}}b^{-\frac{1}{2}}c^{-1} \times a^2b^{\frac{2}{3}}c^{\frac{3}{5}} = a^{\frac{11}{4}}b^{\frac{1}{6}}c^{-\frac{2}{5}}$.

2. $3a^{-2}b^{\frac{2}{3}} \times 2a^{-\frac{4}{5}}b^{\frac{1}{2}}c^2 = 6a^{-\frac{14}{5}}b^{\frac{7}{6}}c^2$.

3. $6a^{-\frac{1}{2}}b^4c^{-m} \times 5a^{\frac{1}{3}}b^{-5}c^n = 30a^{-\frac{1}{6}}b^{-1}c^{n-m}$.

Division of Quantities affected with any Exponents.

235. To divide one monomial by another when both are affected with any exponent whatever, divide the co-efficient of the dividend by that of the divisor, for a new co-efficient: *subtract the exponent of each letter in the divisor from the exponent of the same letter in the dividend, and then annex each letter with its new exponent.*

For, the exponent of each letter in the quotient must be such, that, added to the exponent of the same letter in the divisor, the sum shall be equal to the exponent of the letter in the dividend; hence, the exponent of any letter in the quotient, is equal to the difference between the exponent of that letter in the dividend and in the divisor.

EXAMPLES.

1. $a^{\frac{2}{3}} \div a^{-\frac{3}{4}} = a^{\frac{2}{3}-(-\frac{3}{4})} = a^{\frac{17}{12}}$.

2. $a^{\frac{3}{4}} \div a^{\frac{4}{5}} = a^{\frac{3}{4}-\frac{4}{5}} = a^{-\frac{1}{20}}$.

3. $a^{\frac{2}{5}} \times b^{\frac{3}{4}} \div a^{-\frac{1}{2}}b^{\frac{7}{8}} = a^{\frac{9}{10}}b^{-\frac{1}{8}}$.

4. Divide $32a^{\frac{1}{2}}b^6c^{\frac{5}{2}}$ by $8a^{\frac{1}{6}}b^5c^{-\frac{3}{2}}$. *Ans.* $4a^{\frac{1}{3}}bc^4$.

5. Divide $64a^9b^{\frac{7}{2}}c^{-\frac{3}{5}}$ by $32a^{-9}b^{-\frac{3}{2}}c^{-\frac{3}{5}}$. *Ans.* $2a^{18}b^5$.

Formation of Powers.

236. To form the m^{th} power of a monomial, affected with any exponents whatever, raise the co-efficient to the m^{th} power, and for the exponents, observe the rule given in Art. 220, viz, *multiply the exponent of each letter by the exponent* m *of the power.*

For, to raise a quantity to the m^{th} power, is the same thing as to multiply it by itself $m - 1$ times; therefore, by the rule for multiplication, the exponent of each letter must be added to itself $m - 1$ times, or multiplied by m.

Thus, $\left(a^{\frac{3}{4}}\right)^5 = a^{\frac{15}{4}}$; and $\left(a^{\frac{2}{3}}\right)^3 = a^{\frac{6}{3}} = a^2$;

also, $\left(2a^{-\frac{1}{2}}b^{\frac{3}{4}}\right)^6 = 64a^{-3}b^{\frac{9}{2}}$; and $\left(a^{-\frac{5}{6}}\right)^{12} = a^{-10}$.

What is the m^{th} power of $3a^{\frac{1}{2}}b^{-2}c^2$. *Ans.* $3^m a^{\frac{m}{2}} b^{-2m} c^{2m}$.

Extraction of Roots.

237. To extract the n^{th} root of a monomial, extract the n^{th} root of the co-efficient, and for the new exponents, follow the rule given in Art. 220, viz., *divide the exponent of each letter by the index of the root.*

For, the exponent of each letter in the result should be such, that when multiplied by n, the index of the root to be extracted, the product will be the exponent with which the letter is affected in the proposed monomial; therefore, the exponents in the result must be respectively equal to the quotients arising from the division of the exponents in the proposed monomials, by n, the index of the root. Thus,

$$\sqrt[3]{a^{\frac{2}{3}}} = a^{\frac{2}{9}}; \quad \text{and} \quad \sqrt[4]{a^{\frac{8}{11}}} = a^{\frac{2}{11}};$$

also, $$\sqrt{a^{-\frac{3}{4}}} = a^{-\frac{3}{8}}; \quad \text{and} \quad \sqrt[3]{a^{\frac{3}{5}}b^{-2}} = a^{\frac{1}{5}}b^{-\frac{2}{3}}.$$

The rules for fractional and negative exponents have been easily deduced from the rule for multiplication; but we may give a direct demonstration of them, by going back to the origin of quantities affected with such exponents.

We will demonstrate implicitly, the two preceding rules.

Let it be required to raise $a^{\frac{m}{n}}$ to the $-\frac{r}{s}$ power.

By going back to the origin of these notations, we find that

$$\left(a^{\frac{m}{n}}\right)^{-\frac{r}{s}} = \sqrt[s]{\frac{1}{\left(a^{\frac{m}{n}}\right)^r}} = \sqrt[s]{\frac{1}{(\sqrt[n]{a^m})^r}} = \sqrt[s]{\frac{1}{\sqrt[n]{a^{mr}}}}$$

$$= \sqrt[s]{\sqrt[n]{\frac{1}{a^{mr}}}} = \sqrt[ns]{a^{-mr}} = a^{-\frac{mr}{ns}}; \quad \text{hence,}$$

$$\left(a^{\frac{m}{n}}\right)^{-\frac{r}{s}} = a^{\frac{m}{n} \times -\frac{r}{s}} \text{ is also } = a^{-\frac{mr}{ns}}.$$

REMARK I.—The advantage derived from the use of fractional exponents consists principally in this:—The operations performed upon expressions of this kind require no other rules than those established for the calculus of quantities affected with entire exponents. This calculus is thus reduced to simple operations upon fractions, with which we are already familiar.

REMARK II.—In the resolution of certain questions, we shall be led to consider quantities affected with *incommensurable exponents*. Now, it would seem that the rules just established for commensurable exponents, ought to be demonstrated for the case in which they are incommensurable. But let us observe, that the value of an incommensurable, such as $\sqrt{3}$, $\sqrt[3]{11}$, *may be determined approximatively as near as we please*, so that we can always conceive the incommensurable to be replaced by an exact fraction, which only differs from it by a quantity less than any given quantity; and we apply the rules to the symbol which designates the incommensurable, after substituting the fraction which represents it approximatively.

EXAMPLES.

1. Reduce $\dfrac{2\sqrt{2} \times (3)^{\frac{1}{3}}}{\frac{1}{2}\sqrt{2}}$ to its simplest terms.

Ans. $4\sqrt[3]{3}$.

2. Reduce $\left\{\dfrac{\frac{1}{2}(2)^{\frac{1}{2}}\sqrt[3]{3}}{2\sqrt[4]{2}\,(3)^{\frac{1}{2}}}\right\}^4$ to its simplest terms.

Ans. $\dfrac{1}{384}\sqrt[3]{3}$.

3. Reduce $\sqrt{\left\{\dfrac{(\frac{1}{2})^3 + \sqrt{3\frac{1}{2}}}{2\sqrt{2}\,.\,(\frac{3}{4})^{\frac{1}{2}}}\right\}^{\frac{1}{2}}}$ to its simplest terms.

Ans. $\sqrt[4]{\dfrac{1}{6}\left(\dfrac{1}{8}\sqrt{6} + \sqrt{21}\right)}$.

4. What is the product of

$$a^{\frac{5}{2}} + a^2b^{\frac{1}{3}} + a^{\frac{3}{2}}b^{\frac{2}{3}} + ab + a^{\frac{1}{2}}b^{\frac{4}{3}} + b^{\frac{5}{3}}, \text{ by } a^{\frac{1}{2}} - b^{\frac{1}{3}}.$$

Ans. $a^3 - b^2$.

5. Divide $a^{\frac{7}{3}} - a^2b^{-\frac{2}{3}} - a^{\frac{1}{3}}b + b^{\frac{1}{3}}$, by $a^{\frac{1}{3}} - b^{-\frac{2}{3}}$.

Ans. $a^2 - b$.

Method of Indeterminate Co-efficients.

238. The binomial theorem demonstrated in Art. 203, explains the method of developing into a series any expression of the form $(a + b)^m$, in which m is a whole and positive number.

Algebraists have invented another method of developing algebraic expressions into series, called the method by *indeterminate co-efficients*. This method is more extensive in its applications, can be applied to algebraic expressions of any nature whatever, and indeed, the general case of the binomial theorem may be demonstrated by it.*

Before considering this method, it will be necessary to explain what is meant by the term *function*.

Let $$a = b + c.$$

In this equation, a, b, and c, mutually depend on each other for their values. For,

$$a = b + c, \quad b = a - c, \quad \text{and} \quad c = a - b.$$

The quantity a is said to be a *function* of b and c, b a *function* of a and c, and c a *function* of a and b. And generally, *when one quantity depends for its value on one or more quantities, it is said to be function of each and all the quantities on which it depends.*

239. If we have an equation of the form,

$$A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c. = 0;$$

it is required to find the values of the co-efficients A, B, C, D, E, &c., under the following suppositions:

1st. That no one of the co-efficients is a function of x.

2d. That the series shall be equal to zero, whatever be the number of its terms; and

3d. That it shall be equal to zero, whatever value may be attributed to x.

Now, since the co-efficients are independent of x, their values cannot be affected by any supposition made on the value of x: hence, if they be determined for one value of x, they will be known for all values whatever.

Let us now make

$$x = 0, \quad \text{which gives}$$

$$Bx + Cx^2 + Dx^3 + Ex^4 + \&c. = 0;$$

and consequently, $A = 0$.

If we divide by x, we have

$$B + Cx + Dx^2 + Ex^3 + \&c. = 0,$$

and by again making $x = 0$, we have

$$Cx + Dx^2 + Ex^3 + \&c. = 0;$$

and consequently, $B = 0$.

Dividing again by x, we have

$$C + Dx + Ex^2 + \&c. = 0;$$

and by again making $x = 0$, we obtain

$$Dx + Ex^2 + \&c. = 0,$$

and consequently, $C = 0$;

and by continuing the process we may prove that, *each co-efficient must be separately equal to zero.*

It should be observed, that A may be considered the co-efficient of x^0.

240. The principle demonstrated above, may be enunciated under another form. If we have an equation of the form

$$a + bx + cx^2 + dx^3 + \ .\ .\ . = a' + b'x + c'x^2 + d'x^3 + \ .\ .\ .$$

which is satisfied for any value whatever attributed to x, *the co-efficients of the terms involving the same powers of* x *in the two members, are respectively equal.* For, by transposing all the terms into the first member, the equation will take the form

$$A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c. = 0;$$

whence, $a - a' = 0, \quad b - b' = 0, \quad c - c' = 0 \ .\ .\ .\ .;$

and consequently,

$$a = a', \quad b = b', \quad c = c', \quad d = d' \ .\ .\ .\ .$$

Every equation in which the terms are arranged with reference to a certain letter, and which is satisfied for any value which may be attributed to that letter, is called an *identical equation*, in order to distinguish it from a *common equation*, that is, an equation which can only be satisfied by particular values of the unknown quantity.

241. Let us apply the above principles in developing into a series the expression

$$\frac{a}{a' + b'x}.$$

It is plain, that any expression equal to the above, must contain x, and the quantities a, a', b'. Let us then assume

$$\frac{a}{a' + b'x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c. \; . \; . \; . \; (1),$$

in which the co-efficients A, B, C, D, &c., are functions of a, a', b', and independent of x. These are called, *indeterminate co-efficients*. It is required to find their values in terms of a, a', b', on which they depend.

For this purpose, multiply both members of equation (1) by $a' + b'x$. Arranging the result with reference to the powers of x, and transposing a, we have

$$0 = \left\{ \begin{matrix} Aa' + Ba' \\ -a \;\; + Ab' \end{matrix} \right| x + \begin{matrix} Ca' \\ + Bb' \end{matrix} \left| x^2 + \begin{matrix} Da' \\ + Cb' \end{matrix} \right| x^3 + \begin{matrix} Ea' \\ + Db' \end{matrix} \left| x^4 + \; . \; . \; . \; (2), \right.$$

and since this equation is satisfied for any value of x, we have (Art. 239),

$$Aa' - a = 0, \quad \text{whence,} \quad A = \frac{a}{a'};$$

also, $\quad Ba' + Ab' = 0, \quad$ whence,

$$B = -\frac{Ab'}{a'} = -\frac{a}{a'} \times \frac{b'}{a'} = -\frac{ab'}{a'^2};$$

also, $\quad Ca' + Bb' = 0, \quad$ whence,

$$C = -\frac{Bb'}{a'} = -\left(-\frac{ab'}{a'^2}\right) \times \frac{b'}{a'} = \frac{ab'^2}{a'^3};$$

also $\quad Da' + Cb' = 0, \quad$ whence,

$$D = -\frac{Cb'}{a'} = \frac{ab'^2}{a'^3} \times -\frac{b'}{a'} = -\frac{ab'^3}{a'^4}, \; \&c.$$

It is plain that the terms will be alternately plus and minus, and that the co-efficient of any term is formed by multiplying that of the preceding term by $-\dfrac{b'}{a'}$; therefore, we have

$$\frac{a}{a' + b'x} = \frac{a}{a'} - \frac{ab'}{a'^2}x + \frac{ab'^2}{a'^3}x^2 - \frac{ab'^3}{a'^4}x^3 + \frac{ab'^4}{a'^5}x^4 - \&c.$$

242. The method of *indeterminate co-efficients* requires that we should know the form of the development. The terms of the development are generally arranged according to the ascending powers of x, commencing with the power x^0; sometimes, however, this form is not applicable, in which case, the calculus detects the error in the supposition.

For example, develop the expression $\dfrac{1}{3x - x^2}$.

Let us suppose that

$$\frac{1}{3x - x^2} = A + Bx + Cx^2 + Dx^3 + \ldots,$$

whence, by reducing to entire terms, and arranging with reference to x,

$$0 = -1 + 3Ax + \begin{array}{c|} 3B \\ -A \end{array}\, x^2 + \begin{array}{c|} 3C \\ -B \end{array}\, x^3 + \begin{array}{c|} 3D \\ -C \end{array}\, x^4 + \ldots,$$

whence (Art. 239),

$$-1 = 0, \quad 3A = 0, \quad 3B - A = 0 \ldots.$$

Now, the first equation, $-1 = 0$, is absurd, and indicates that the above form will not develop the expression $\dfrac{1}{3x - x^2}$. But if we put the expression under the form $\dfrac{1}{x} \times \dfrac{1}{3 - x}$, and make

$$\frac{1}{3 - x} = A + Bx + Cx^2 + Dx^3 + \ldots,$$

we shall have, after the reductions are made,

$$0 = \left\{ \begin{array}{c|} 3A + 3B \\ -1 - A \end{array}\, x + \begin{array}{c|} 3C \\ -B \end{array}\, x^2 + \begin{array}{c|} 3D \\ -C \end{array}\, x^3 + \ldots, \right.$$

which gives the equations

$$3A - 1 = 0, \quad 3B - A = 0, \quad 3C - B = 0 \ldots,$$

whence, $\quad A = \dfrac{1}{3}, \quad B = \dfrac{1}{9}, \quad C = \dfrac{1}{27}, \quad D = \dfrac{1}{81} \ldots$

Therefore,

$$\frac{1}{3x - x^2} = \frac{1}{x}\left(\frac{1}{3} + \frac{1}{9}x + \frac{1}{27}x^2 + \frac{1}{81}x^3 + \ldots\right),$$

or

$$= \frac{1}{3}x^{-1} + \frac{1}{9}x^0 + \frac{1}{27}x + \frac{1}{81}x^2 + \ldots;$$

that is, the development contains a term with a negative exponent.

Recurring Series.

243. The development of algebraic fractions by the method of indeterminate co-efficients, gives rise to certain series, called *recurring series.*

A recurring series is the development of a rational fraction involving x, *made according to a fixed law, and containing the ascending powers of* x *in its different terms.*

It has been shown in Art. 241, that the expression

$$\frac{a}{a'+b'x}=\frac{a}{a'}-\frac{ab'}{a'^2}x+\frac{ab'^2}{a'^3}x^2-\frac{ab'^3}{a'^4}x^3+\ldots,$$

in which each term is formed by multiplying that which precedes it by $-\frac{b'}{a'}x$.

This property of determining one term of the development from those which precede, is not peculiar to the proposed fraction; it belongs to all rational algebraic fractions, and may be thus expressed; viz., *Every rational fraction involving* x, *may be developed into a series of terms, each of which is equal to the algebraic sum of the products which arise from multiplying certain terms of a particular expression, by certain of the preceding terms of the series.*

The particular expression, from which any term of the series may be found, when the preceding terms are known, is called the *scale* of the series; and that from which the co-efficient may be formed, the *scale* of the co-efficients.

In the preceding series, the *scale* is $-\frac{b'}{a'}x$, and the series is called *a recurring series of the first order;* and $-\frac{b'}{a'}$ is the scale of the co-efficients.

Let it be required to develop $\frac{a+bx}{a'+b'x+c'x^2}$ into a series.

Assume

$$\frac{a+bx}{a'+b'x+c'x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\ldots$$

reducing to entire terms, and transposing, we have

$$0=\left\{\begin{array}{r|r|r|r|l} Aa'+Ba' & x+Ca' & x^2+Da' & x^3+Ea' & x^4+\ldots \\ -a+Ab' & +Bb' & +Cb' & +Db' & \\ -\ b & +Ac' & +Bc' & +Cc' & \end{array}\right.$$

which gives the equations

$$Aa' - a = 0, \quad \text{or} \quad A = \frac{a}{a'},$$

$$Ba' + Ab' - b = 0, \quad \text{whence,} \quad B = -\frac{b'}{a'}A + \frac{b}{a'},$$

$$Ca' + Bb' + Ac' = 0, \quad \text{whence,} \quad C = -\frac{b'}{a'}B - \frac{c'}{a'}A,$$

$$Da' + Cb' + Bc' = 0, \quad \text{whence,} \quad D = -\frac{b'}{a'}C - \frac{c'}{a'}B,$$

$$Ea' + Db' + Cc' = 0, \quad \text{whence,} \quad E = -\frac{b'}{a'}D - \frac{c'}{a'}C.$$

. . . .

. . . .

From which we see, that the first two co-efficients are not obtained by any law; but commencing at the third, each co-efficient is formed by multiplying the two which precede it, respectively, by $-\frac{b'}{a'}$ and $-\frac{c'}{a'}$, viz., that which immediately precedes the required co-efficient by $-\frac{b'}{a'}$, that which precedes it two terms by $-\frac{c'}{a'}$, and taking the algebraic sum of the products. Hence,

$$\left(-\frac{b'}{a'} - \frac{c'}{a'}\right)$$

is the *scale of the co-efficients.*

From this law of the formation of the co-efficients, it follows that the third term of the series, Cx^2, is equal to

$$-\frac{b'}{a'}Bx^2 - \frac{c'}{a'}Ax^2;$$

or

$$-\frac{b'}{a'}x \,.\, Bx - \frac{c'}{a'}x^2 \,.\, A.$$

In like manner, we have for Dx^3,

$$-\frac{b'}{a'}Cx^3 - \frac{c'}{a'}Bx^3;$$

or

$$-\frac{b'}{a'}x \,.\, Cx^2 - \frac{c'}{a'}x^2 \,.\, Bx.$$

Hence, each term of the required series, commencing at the third, is obtained by multiplying the two terms which precede,

respectively by

$$-\frac{b'}{a'}x - \frac{c'}{a'}x^2,$$

and taking the sum of the products: hence, this last expression is the *scale* of the series.

Recurring series are divided into orders, and the order is estimated by the number of terms in the *scale* which involve x.

Thus, the expression $\frac{a}{a' + b'x}$ gives a recurring series of *the first order*, the scale of which is $-\frac{b'}{a'}x$.

The expression $\frac{a + bx}{a' + b'x + c'x^2}$ gives a recurring series of *the second order*, of which the scale is

$$\left(-\frac{b'}{a'}x, \quad -\frac{c'}{a'}x^2\right).$$

The series obtained in the preceding Art. is of the second order.

In general, an expression of the form

$$\frac{a + bx + cx^2 + \ldots kx^{n-1}}{a' + b'x + c'x^2 + \ldots k'x^n}.$$

gives a recurring series *of the* nth *order*, the scale of which is

$$\left(-\frac{b'}{a'}x, \quad -\frac{c'}{a'}x^2 \ldots -\frac{k'}{a'}x^n\right).$$

REMARK.—It is here supposed that the degree of x in the numerator is less than it is in the denominator. If it was not, it would first be necessary to perform the division, arranging the quantities with reference to x, which would give an entire quotient, plus a fraction similar to the above.

Thus, in the expression $\frac{1 - x - 3x^2 + 4x^3 + x^4}{2 - 5x + 3x^2 - x^3}$, gives

$$\begin{array}{l|l} x^4 + 4x^3 - 3x^2 - x + 1 & -x^3 + 3x^2 - 5x + 2 \\ \hline \quad + 7x^3 - 8x^2 + x & -x \;\; - 7 \\ \cline{1-1} \qquad + 13x^2 - 34x + 15. & \end{array}$$

Performing the division, we find the quotient to be $-x - 7$, plus the fraction

$$\frac{13x^2 - 34x + 15}{-x^3 + 3x^2 - 5x + 2} = \frac{15 - 34x + 13x^2}{2 - 5x + 3x^2 - x^3}.$$

Demonstration of the Binomial Theorem for any Exponent.

244. It has been shown (Art. 61), that any expression of the form

$$x^m - y^m, \text{ is exactly divisible by } x - y,$$

when m is a positive whole number. That is,

$$\frac{x^m - y^m}{x - y} = x^{m-1} + x^{m-2}y + x^{m-3}y^2 + \ldots + xy^{m-2} + y^{m-1}.$$

The number of terms in the second member is equal to m; and if we suppose $x = y$, each term will become equal to x^{m-1}: hence,

$$\frac{x^m - x^m}{x - x} = mx^{m-1}.$$

We propose to prove that the quotient will have the same form when m is negative, and also, when m is a positive or negative fraction.

First, when m is a whole number, and negative.

Let n be a positive whole number, and numerically equal to m. Then,

$$m = -n.$$

By observing that

$$-x^{-n}y^{-n} \times (x^n - y^n) = x^{-n} - y^{-n},$$

we have

$$\frac{x^{-n} - y^{-n}}{x - y} = -x^{-n}y^{-n} \times \frac{(x^n - y^n)}{x - y} = -x^{-2n}nx^{n-1} = -nx^{-n-1},$$

after making $y = x$; and by restoring m,

$$\frac{x^m - x^m}{x - x} = mx^{m-1}, \quad m \text{ being a negative number.}$$

Second, let m be a positive fraction, or $m = \frac{p}{q}$.

Let $x^{\frac{1}{q}} = v$, whence, $x^{\frac{p}{q}} = v^p$, and $x = v^q$;

and $y^{\frac{1}{q}} = u$, whence, $y^{\frac{p}{q}} = u^p$, and $y = u^q$.

Hence,
$$\frac{x^{\frac{p}{q}} - y^{\frac{p}{q}}}{x - y} = \frac{v^p - u^p}{v^q - u^q} = \frac{\dfrac{v^p - u^p}{v - u}}{\dfrac{v^q - u^q}{v - u}}.$$

If we suppose $x = y$, we have $v = u$; and since p and q are positive whole numbers, we have

$$\frac{x^{\frac{p}{q}} - x^{\frac{p}{q}}}{x - x} = \frac{pv^{p-1}}{qv^{q-1}} = \frac{p}{q}v^{p-q} = \frac{p}{q}x^{\frac{p}{q}-1},$$

after substituting for v its value $x^{\frac{1}{q}}$.

If we restore m, we have

$$\frac{x^m - x^m}{x - x} = mx^{m-1}, \quad m \text{ being a positive fraction.}$$

Third, let m *be a negative fraction, or* $m = -\frac{p}{q}$.

Let $\quad x^{\frac{1}{q}} = v, \quad$ whence, $\quad x^{-\frac{p}{q}} = v^{-p}, \quad$ and $\quad x = v^q$;

and $\quad y^{\frac{1}{q}} = u, \quad$ whence, $\quad y^{-\frac{p}{q}} = u^{-p}, \quad$ and $\quad y = u^q$.

Hence, $$\frac{x^{-\frac{p}{q}} - y^{-\frac{p}{q}}}{x - y} = \frac{v^{-p} - u^{-p}}{v^q - u^q} = \frac{\dfrac{v^{-p} - u^{-p}}{v - u}}{\dfrac{v^q - u^q}{v - u}}.$$

But since $-p$ is a negative integer, and q a positive integer, we have from what has preceded, after making $x = y$, which gives $u = v$,

$$\frac{x^{-\frac{p}{q}} - x^{-\frac{p}{q}}}{x - x} = \frac{-pv^{-p-1}}{qv^{q-1}} = -\frac{p}{q}v^{-p-q};$$

and substituting for v its value $x^{\frac{1}{q}}$ we have

$$\frac{x^{-\frac{p}{q}} - x^{-\frac{p}{q}}}{x - x} = -pv^{-p-q} = -\frac{p}{q}x^{-\frac{p}{q}-1};$$

and restoring m, we have

$$\frac{x^m - x^m}{x - x} = mx^{m-1}, \quad m \text{ being a negative fraction.}$$

245. We are now prepared to find a general formula for the development of the binomial $(a + b)^m$, *in which the exponent* m *is positive or negative, and either integral or fractional.*

In order to simplify the process, let us place the binomial

under the form

$$(a+b)^m = [a\left(1+\frac{b}{a}\right)]^m = a^m\left(1+\frac{b}{a}\right)^m \text{ (Art. 220).}$$

If we find the development of $\left(1+\frac{b}{a}\right)^m$, and then multiply it by a^m, the product will be the development of $(a+b)^m$.

In order further to simplify the expression, let us make

$$\frac{b}{a} = x;$$

then, the binomial to be developed will be of the form

$$(1+x)^m.$$

As this development must be expressed in terms of x, and known quantities dependent for their values on 1 and m, we may assume

$$(1+x)^m = A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c. \quad . \; . \; . \; (1),$$

in which the co-efficients A, B, C, &c., are independent of x, and functions of 1 and m.

Now, since this equation is true for any value of x, if we make $x = 0$, we have

$$(1)^m = A = 1.$$

Substituting this value in equation (1), we have

$$(1+x)^m = 1 + Bx + Cx^2 + Dx^3 + Ex^4 + \&c. \quad . \; . \; . \; (2).$$

Since the form of the above development will not be changed by placing y for x, we may write

$$(1+y)^m = 1 + By + Cy^2 + Dy^3 + Ey^4 + \&c. \quad . \; . \; . \; (3).$$

Subtracting equation (2) from (3), and dividing both members by $x - y$, we have

$$\frac{(1+x)^m-(1+y)^m}{x-y} = B\frac{(x-y)}{x-y} + C\frac{(x^2-y^2)}{x-y} + D\frac{(x^3-y^3)}{x-y} + \&c. \ldots (4).$$

Make $1 + x = v$, and $1 + y = u$; whence, $x - y = v - u$.

Substituting these values in the first member of equation (4), and we have

$$\frac{v^m - u^m}{v-u} = B\frac{(x-y)}{x-y} + C\frac{(x^2-y^2)}{x-y} + D\frac{(x^3-y^3)}{x-y} + \&c. \ldots (5).$$

If now, we make

$$x = y, \quad \text{whence,} \quad v = u,$$

we have, from Art. 244,

$$\frac{v^m - u^m}{v - u} = \frac{v^m - v^m}{v - v} = mv^{m-1} = m(1 + x)^{m-1};$$

while the quotients in the second member become, respectively,

$$\frac{x - y}{x - y} = 1, \quad \frac{x^2 - y^2}{x - y} = 2x^{2-1} = 2x;$$

$$\frac{x^3 - y^3}{x - y} = 3x^{3-1} = 3x^2; \quad \frac{x^4 - y^4}{x - y} = 4x^{4-1} = 4x^3; \text{ \&c.}$$

Substituting these values in equation (5), and we have

$$m(1 + x)^{m-1} = + B + 2Cx + 3Dx^2 + 4Ex^3 + \text{\&c.} \; . \; . \; . \; . \; (6).$$

Multiplying both members of this equation by $1 + x$, and arranging the second member with reference to x, and we have

$$m(1 + x)^m = \begin{array}{c|c|c|l} B + 2C & x + 3D & x^2 + 4E & x^3 + \text{\&c.} \\ + B & + 2C & + 3D & \end{array}$$

If we now multiply equation (2) by m, we have

$$m(1 + x)^m = m + mBx + mCx^2 + mDx^3 + mEx^4 + \text{\&c.}$$

If we place the second member of the last two equations equal to each other, we shall obtain an identical equation. Then, placing the co-efficients of the like powers of x equal to each other (Art. 240), we have

$$B = m, \quad \text{whence,} \quad B = \frac{m}{1};$$

$$2C + B = mB, \quad \text{whence,} \quad C = \frac{B(m-1)}{2} = \frac{m(m-1)}{1 \,.\, 2};$$

$$3D + 2C = mC, \quad \text{whence,} \quad D = \frac{C(m-2)}{3} = \frac{m(m-1)(m-2)}{1 \,.\, 2 \,.\, 3};$$

$$4E + 3D = mD, \quad \text{whence,} \quad E = \frac{D(m-3)}{4} = \frac{m(m-1)(m-2)(m-3)}{1 \,.\, 2 \,.\, 3 \,.\, 4}.$$

$$\text{\&c.,} \qquad \qquad \text{\&c.}$$

Substituting these values of A, B, C, D, &c., in equation (2), we obtain

$$(1 + x)^m = 1 + mx + \frac{m(m-1)}{1 \,.\, 2}x^2 + \frac{m(m-1)(m-2)}{1 \,.\, 2 \,.\, 3}x^3 + \frac{m(m-1)(m-2)(m-3)}{1 \,.\, 2 \,.\, 3 \,.\, 4}x^4 + \text{\&c.}$$

If we now replace x by its value $\frac{b}{a}$, we have

$$\left(1+\frac{b}{a}\right)^m = 1 + m\frac{b}{a} + \frac{m(m-1)}{1 \;.\; 2}\frac{b^2}{a^2} + \frac{m(m-1)(m-2)}{1 \;.\; 2 \;.\; 3}\frac{b^3}{a^3} + \&c.$$

Finally, multiplying by a^m, we obtain

$$(a+b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{1 \;.\; 2}a^{m-2}b^2$$
$$+ \frac{m(m-1)(m-2)}{1 \;.\; 2 \;.\; 3}a^{m-3}b^3 + \&c.;$$

a development which is of the same form as the one obtained in Art. 203, under the supposition of m being a positive and whole number.

Applications of the Binomial Theorem.

If in the formula $(x+a)^m =$

$$x^m\left(1 + m\,.\,\frac{a}{x} + m\,.\,\frac{m-1}{2}\,.\,\frac{a^2}{x^2} + m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}\,.\,\frac{a^3}{x^3} + \ldots\right)$$

we make $m = \frac{1}{n}$, it becomes $(x+a)^{\frac{1}{n}}$ or $\sqrt[n]{x+a} =$

$$x^{\frac{1}{n}}\left(1 + \frac{1}{n}\,.\,\frac{a}{x} + \frac{1}{n}\,.\,\frac{\frac{1}{n}-1}{2}\,.\,\frac{a^2}{x^2} + \frac{1}{n}\,.\,\frac{\frac{1}{n}-1}{2}\,.\,\frac{\frac{1}{n}-2}{3}\,.\,\frac{a^3}{x^3} + \ldots\right)$$

or, reducing, $\sqrt[n]{x+a} =$

$$x^{\frac{1}{n}}\left(1 + \frac{1}{n}\,.\,\frac{a}{x} - \frac{1}{n}\,.\,\frac{n-1}{2n}\,.\,\frac{a^2}{x^2} + \frac{1}{n}\,.\,\frac{n-1}{2n}\,.\,\frac{2n-1}{3n}\,.\,\frac{a^3}{x^3} - \ldots\ldots\right)$$

The fifth term can be found by multiplying the fourth by $\frac{3n-1}{4n}$ and by $\frac{a}{x}$, then changing the sign of the result, and so on.

REMARK.—If in this formula, we make $n = 2$, $n = 3$, $n = 4$, &c., the development will become the approximate square root, cube root, fourth root, &c., of the binomial $(x+a)$; and by assigning values at pleasure to x and a as well as to n, we can find any root whatever of any binomial. If n is negative, or fractional, there will be no limit to the number of terms to which the series may be carried. Such a series is called an *infinite series.*

The binomial formula also serves to develop algebraic expressions into series.

Take, for example, the expression $\frac{1}{1-z}$, we have

$$\frac{1}{1-z}=(1-z)^{-1}.$$

In the binomial formula, make $m=-1$, $x=1$, and $a=-z$, it becomes

$$(1-z)^{-1}=1-1\,.\,(-z)-1\,.\,\frac{-1-1}{2}\,.\,(-z)^2$$
$$-1\,.\,\frac{-1-1}{2}\,.\,\frac{-1-2}{3}\,.\,(-z)^3-\ .\ .\ .$$

or, performing the operations, and observing that each term is composed of an even number of factors affected with the sign $-$

$$(1-z)^{-1}=\frac{1}{1-z}=1+z+z^2+z^3+z^4+z^5+\ .\ .\ .\ .$$

The same result will be obtained by applying the rules for division (Art. 55).

	1	$1-z$
1st remainder -	$+z$	$1+z+z^2+z^3+z^4+\ .\ .\ .$
2d - - - - -	$+z^2$	
3d - - - - -	$+z^3$	
4th - - - - -	$+x^4$	
	$+\ .\ .\ .$	

Again, take the expression $\frac{2}{(1-z)^3}$, or $2(1-z)^{-3}$.

We have $2(1-z)^{-3}=$

$$2[1-3\,.(-z)-3\,.\frac{-3-1}{2}.(-z)^2-3\,.\frac{-3-1}{2}\,.\frac{-3-2}{3}.(-z)^3-.];$$

or $\quad 2(1-z)^{-3}=2(1+3z+6z^2+10z^3+15z^4+\ .\ .\ .)$

To develop the expression $\sqrt[3]{2z-z^2}$, which reduces to

$\sqrt[3]{2z}\left(1-\frac{z}{2}\right)^{\frac{1}{3}}$, we first find

$$\left(1-\frac{z}{2}\right)^{\frac{1}{3}}=1+\frac{1}{3}\left(-\frac{z}{2}\right)+\frac{1}{3}\,.\,\frac{\frac{1}{3}-1}{2}\,.\left(-\frac{z}{2}\right)^2+\ .\ .\ .$$
$$=1-\frac{1}{6}z-\frac{1}{36}z^2-\frac{5}{648}z^3-\ .\ .\ .;$$

hence $\sqrt[3]{2z - z^2} = \sqrt[3]{2z}\left(1 - \frac{1}{6}z - \frac{1}{36}z^2 - \frac{5}{648}z^3 -, \text{\&c.}\right)$

EXAMPLES.

1. To find the value of $\frac{1}{(a+b)^2} = (a+b)^{-2}$ in an infinite series.

2. To find the value of $\frac{r^2}{r+x}$ in an infinite series.

$$Ans.\ r - x + \frac{x^2}{r} - \frac{x^3}{r^2} + \frac{x^4}{r^3}, \text{ \&c.}$$

3. Required the square root of $\frac{a^2+x^2}{a^2-x^2}$ in an infinite series.

$$Ans.\ 1 + \frac{x^2}{a^2} + \frac{x^4}{2a^4} + \frac{x^6}{2a^6}, \text{ \&c.}$$

4. Required the cube root of $\frac{a^2}{(a^2+x^2)^2}$ in an infinite series.

$$Ans.\ \frac{1}{a^{\frac{4}{3}}} \times \left(1 - \frac{2x^2}{3a^2} + \frac{5x^4}{9a^4} - \frac{40x^6}{81a^6}, \text{ \&c.}\right)$$

REMARK.—When the terms of a series go on decreasing in value, the series is called a *decreasing* series; and when they go on increasing in value, it is called an *increasing* series.

A *converging series* is one in which the greater the number of terms taken, the nearer will their sum approximate to the true value of the entire series. When the terms of a decreasing and converging series are *alternately positive and negative*, we can, by taking a given number, determine the *degree of approximation*.

For, let $a - b + c - d + e - f + \ldots$, &c., be a decreasing series, $b, c, d, \ldots$ being positive quantities, and let x denote the true value of this series. Then, if n denote any number of terms, the value of x will be found between the sum of the n and $n+1$ terms.

For, take any two consecutive sums,

$$a - b + c - d + e - f, \text{ and } a - b + c - d + e - f + g.$$

In the first, the terms which follow $-f$, are

$$+g - h, \ +k - l + \ldots;$$

but since the series is decreasing, the differences of the consecutive terms $g - h$, $k - l$, . . . are positive numbers; therefore, in order to obtain the complete value of x, a positive number must be added to the sum $a - b + c - d + e - f$. Hence, we have

$$a - b + c - d + e - f < x.$$

In the second series, the terms which follow $+g$, are $-h + k$, $-l + m$ . . . . Now, the differences $-h + k$, $-l + m$ . . ., of the consecutive terms, are negative; therefore, in order to obtain the sum of the series, a negative quantity must be added to

$$a - b + c - d + e - f + g,$$

or, in other words, it is necessary to diminish it. Consequently,

$$a - b + c - d + e - f + g > x.$$

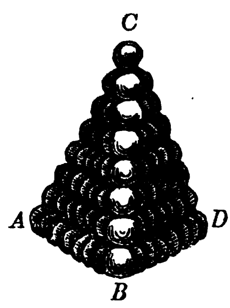
Therefore, x *is comprehended between the sum of the* n *and* n + 1 *terms.*

But the difference between these two sums is equal to g; and since x is comprised between them, their difference g must be greater than the difference between x and either of them; hence, the *error committed by taking* n *terms,* a − b + c − d + e − f, *of the series, for the value of* x, *is numerically less than the following term.*

Summation of Series.

246. An interesting, and at the same time useful application of the principles involved in the summation of series, is found in determining the number of balls or shells contained in a given pile.

Let ABC be a triangular pile of balls, having eight balls on each of the three equal lines, AB, BD, and AD, and also, eight balls in height along the line CB.



Now, the proposed pile consists of 8 horizontal courses, and the number of shot in each course, is the sum of an arithmetical series of which the first term is 1, the last term the number of courses from C, and the number of terms, also the number of courses from C. There-

fore we have

1st course is equal to	$(1 + 1) \times \frac{1}{2} =$	1;
2d " "	$(1 + 2) \times 1 =$	3;
3d " "	$(1 + 3) \times 1\frac{1}{2} =$	6;
4th " "	$(1 + 4) \times 2 =$	10;
5th " "	$(1 + 5) \times 2\frac{1}{2} =$	15;
6th " "	$(1 + 6) \times 3 =$	21;
7th " "	$(1 + 7) \times 3\frac{1}{2} =$	28;
8th " "	$(1 + 8) \times 4 =$	36.

Hence, the number of shot in the pile will be equal to the sum of the series

$$1, \ 3, \ 6, \ 10, \ 15, \ 21, \ 28, \ 36;$$

in which any term is found by *adding* 1 *to the number of the term and multiplying the sum by half the number of terms.*

Thus, if we suppose the horizontal layers to be continued down, and denote the number of any layer from the top by n, we shall have

$$1, \ 3, \ 6, \ 10, \ 15, \ 21, \ \ldots \ \frac{n(n+1)}{2};$$

and the sum of this series will express the number of balls in a triangular pile, of which n denotes the number in either of the bottom rows.

If the general term of any increasing series of numbers involves n to the m^{th} degree, the sum of the series will not involve n to a higher degree than $(m + 1)$. For, the sum of such series cannot exceed n times the general term, and hence, cannot involve n to a higher degree than $m + 1$. Let us therefore assume

$$1 + 3 + 6 + 10 + 15 \ \ldots \ \frac{n(n+1)}{2} = A + Bn + Cn^2 + Dn^3,$$

in which the co-efficients A, B, C, and D, are not functions of n. In order that these co-efficients may be determined, we must find four independent equations involving them. If we make

$n = 1$, we have $A + B + C + D = 1 \qquad = 1 \quad (1)$,

$n = 2$, gives $A + 2B + 4C + 8D = 1 + 3 \qquad = 4 \quad (2)$,

$n = 3$, " $A + 3B + 9C + 27D = 1 + 3 + 6 \qquad = 10 \quad (3)$,

$n = 4$, " $A + 4B + 16C + 64D = 1 + 3 + 6 + 10 = 20 \quad (4)$.

Now, by a series of subtractions we have

Equation (2)—(1), gives $B + 3C + 7D = 3$ . . . (5),

" (3)—(2), " $B + 5C + 19D = 6$ . . . (6),

" (4)—(3), " $B + 7C + 37D = 10$ . . . (7),

" (6)—(5), " $2C + 12D = 3$ . . . (8),

" (7)—(6), " $2C + 18D = 4$ . . . (9),

" (9)—(8), " $6D = 1$; hence, $D = \frac{1}{6}$;

also, $2C + 18D = 4$, gives $C = \frac{1}{2}$;

$B + 7C + 37D = 10$, " $B = \frac{1}{3}$;

$A + B + C + D = 1$, " $A = 0$.

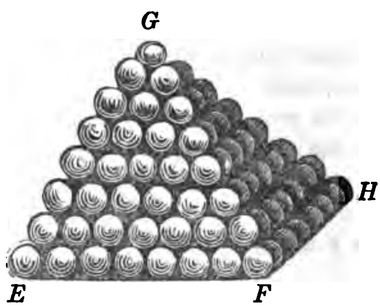
Hence,

$$1 + 3 + 6 + 10 + 15 + \ldots \frac{n(n+1)}{2} = \tfrac{1}{3}n + \tfrac{1}{2}n^2 + \tfrac{1}{6}n^3$$

$$= \frac{n}{6}(2 + 3n + n^2)$$

$$= \frac{n(n+1)(n+2)}{1 \;.\; 2 \;.\; 3}.$$

Let us suppose that we have a pile of balls whose base is a square, two sides of which, EF, FH, are seen in the figure, and that it terminated by a single ball at G.



Now, the number of balls in the upper course will be expressed by 1^2, in the second course by 2^2, in the third course by 3^2, &c. Hence, the series

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots n^2,$$

will express the number of balls in a square pile, of which the number of courses, and consequently the number of balls in one of the lower rows is, n.

To find the sum of this series, assume

$$1 + 4 + 9 + 16 + \ldots n^2 = A + Bn + Cn^2 + Dn^3,$$

from which we find

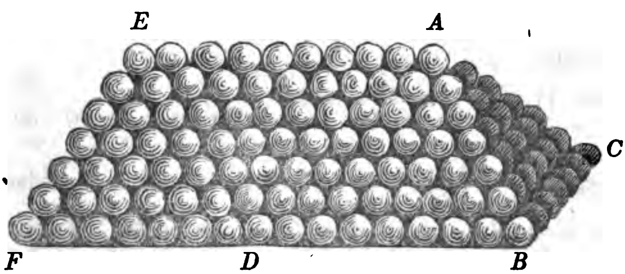
$$\begin{aligned}
A + B + C + D &= 1 &&= 1, \\
A + 2B + 4C + 8D &= 1 + 4 &&= 5; \\
A + 3B + 9C + 27D &= 1 + 4 + 9 &&= 14; \\
A + 4B + 16C + 64D &= 1 + 4 + 9 + 16 &&= 30;
\end{aligned}$$

and from these four equations, we find, by continued subtractions,

$$D = \tfrac{1}{3}, \quad C = \tfrac{1}{2}, \quad B = \tfrac{1}{6}, \quad \text{and} \quad A = 0; \text{ hence,}$$

$$\begin{aligned}
1 + 4 + 9 + 16 + 25 \ldots\ldots n^2 &= \tfrac{1}{6}n + \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 \\
&= \frac{n}{6}(2n^2 + 3n + 1) \\
&= \frac{n(n+1)(2n+1)}{1 \,.\, 2 \,.\, 3}.
\end{aligned}$$

Let us now suppose that we have a rectangular or oblong pile of shot, as represented in the figure below.



Suppose we take off from the oblong pile the square pile *EFD*. We then see that the oblong pile may be formed by adding to the square pile a series of triangular strata, each containing as many balls as are contained in one of the faces of the square pile; and the number of the triangular strata will be one less than the number of balls in the top row. Therefore, if n denote the number of horizontal courses, the number of balls in one triangular strata will be expressed by $\dfrac{n(n+1)}{2}$; and if $m+1$ denotes the whole number of balls in the top row, the number of triangular strata will be denoted by m; and the number of balls in all these strata by

$$\frac{n(n+1)}{2} \times m.$$

But since the number of balls in a square pile, whose side contains n balls is

$$\frac{n(n+1)(2n+1)}{1\ .\ 2\ .\ 3},$$

the number of balls in an oblong pile, whose top row contains $m+1$ balls, and depth n balls, will be expressed by

$$\frac{n(n+1)(2n+1)}{1\ .\ 2\ .\ 3} + \frac{n(n+1)}{2} \times m$$

$$= \frac{n(n+1)}{2} \times \frac{(1+2n+3m)}{3}.$$

If we denote the general sum by S, we shall have the following formulas for the number of shot in each pile.

Triangular, $S = \frac{n(n+1)}{2} . \frac{(n+2)}{3} = \frac{1}{3} . \frac{n(n+1)}{2} . (n+1+1).$

Square, $S = \frac{n(n+1)}{2} . \frac{(2n+1)}{3} = \frac{1}{3} . \frac{n(n+1)}{2} . (n+n+1).$

Rectangular,

$$S = \frac{n(n+1)}{2} . \frac{(2n+1+3m)}{3} = \frac{1}{3} . \frac{n(n+1)}{2} . [(n+m)+(m+n)+(m+1)].$$

Now, since $\frac{n(n+1)}{2}$ is the number of balls in the triangular face of each pile, and the other factor, the number of balls in the longest line of the base plus the number in the side of the base opposite, plus the parallel top row, we have the following

RULE.

Add to the number of balls in the longest line of the base, the number in the parallel side opposite, and also the number in the top parallel row; then multiply this sum by one third the number in triangular face.

EXAMPLES.

1. How many balls in a triangular pile of 15 courses?

Ans. 680.

2. How many balls in a square pile of 14 courses? and how many will remain after 5 courses are removed?

Ans. 1015 and 960.

3. In an oblong pile the length and breadth at bottom are respectively 60 and 30: how many balls does it contain? *Ans.* 23405.

4. In an incomplete rectangular pile, the length and breadth at bottom are respectively 46 and 20, and the length and breadth at top 35 and 9: how many balls does it contain? *Ans.* 7190.

Summation of infinite Series.

247. An infinite series is a succession of terms unlimited in number, and derived from each other according to some fixed and known law.

The summation of a series consists in finding an expression of a finite value, equivalent to the sum of all its terms.

Different series are governed by different laws, and the methods of finding the sum of the terms which are applicable to one class, will not apply universally. A great variety of useful series may be summed by the following formula:

Assume $$\frac{q}{n} - \frac{q}{n+p} = \frac{pq}{n(n+p)}:$$

then, $$\frac{q}{n(n+p)} = \frac{1}{p}\left(\frac{q}{n} - \frac{q}{n+p}\right).$$

If now, by attributing known values to p and q, and different values in succession to n, the expression $\frac{q}{n(n+p)}$ shall represent a given series; then, the sum of this series will be equal to $\frac{1}{p}$ multiplied by the difference between the two new series of which $\frac{q}{n}$ and $\frac{q}{n+p}$ are the general terms. Hence, if the *difference* of the sums of these series be known, and the value of $\frac{1}{p}$ be known, we can find the value of the series $\frac{q}{n(n+p)}$, by the formula $S = \frac{1}{p}(s' - s'')$ even if we do not know the value of the new series $\frac{q}{n}$ and $\frac{q}{n+p}$.

EXAMPLES.

1. Required the sum of the series

$$\frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.3.4} + \frac{1}{1.4.5} + \text{\&c., to infinity.}$$

We see that if we make $q = 1$, and $p = 1$, and $n = 1, 2, 3, 4$, &c., in succession, that the first member of the formula,

$$\frac{q}{n(n+p)},$$

will, in succession, represent each term of the series; while under the same supposition, the second member will become, for n terms of the series,

$$\left\{\begin{array}{l} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} \ldots \frac{1}{n} \\ -\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4} \ldots \frac{1}{n+1}\right) \end{array}\right\} = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

If now, we suppose $n = \infty$, the value of the sum of the series will become equal to 1.

2. Required the sum of n terms of the series

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\frac{1}{9.11}+\text{\&c., to infinity.}$$

To adapt the formula to this series, we make $q = 1$, $p = 2$, and $n = 1, 3, 5, 7$, &c.; we then have, for the sum of n terms,

$$\left\{\begin{array}{l} 1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7} \ldots \frac{1}{2n-1} \\ -\left(\frac{1}{3}+\frac{1}{5}+\frac{1}{7} \ldots \frac{1}{2n-1}+\frac{1}{2n+1}\right) \end{array}\right\} = 1 - \frac{1}{2n+1}$$

$$= \frac{2n}{2n+1}, \text{ and } \frac{1}{p} \text{ of this sum } = \frac{n}{2n+1}.$$

If now, we suppose $n = \infty$, the value of the series becomes equal to one half.

3. Required the sum of n terms of the series

$$\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}+\text{\&c., to infinity.}$$

Here $p = 3$, $q = 1$, $n = 1, 2, 3, 4$, &c.: hence,

$$\frac{1}{3}\left\{\begin{array}{l} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots \frac{1}{n} \\ -\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6} \ldots \frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right) \end{array}\right\}$$

$$= \frac{1}{3}\left[\frac{11}{6} - \left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right)\right] = \frac{11}{18},$$

when $n = \infty$.

4. Find the sum of n terms of the series

$$\frac{2}{3\,.\,5} - \frac{3}{5\,.\,7} + \frac{4}{7\,.\,9} - \frac{5}{9\,.\,11} + \frac{6}{11\,.\,13} + \text{\&c., to infinity}$$

$$\left\{\begin{array}{l} \frac{2}{3} - \frac{3}{5} + \frac{4}{7} - \frac{5}{9} + \;.\;.\;.\;.\; \mp \frac{n+1}{2n+1} \\ \quad - \left(\frac{2}{5} - \frac{3}{7} + \frac{4}{9} - \;.\;.\;.\;.\; \pm \frac{n}{2n+1} \mp \frac{n+1}{2n+3}\right. \end{array}\right\}$$

which becomes,

$$\frac{2}{3} \pm \frac{n+1}{2n+3} - (1 - 1 + 1 - \;.\;.\;.\;.\;.\;.\; \pm 1).$$

If the number of terms used is even, the upper sign will apply, the quantity within the parenthesis will become $+1$, and the sum of the n terms before dividing by p, is

$$-\frac{1}{3} + \frac{n+1}{2n+3} = \frac{1}{6}, \text{ when } n = \infty.$$

If n is odd, the lower sign is used, and the quantity within the parenthesis reduces to zero, and we have

$$\frac{2}{3} - \frac{n+1}{2n+3} = \frac{1}{6}, \text{ when } n = \infty.$$

Then, since $p = 2$, the sum of the series when $n = \infty$, is $\frac{1}{12}$.

5. Required the sum of the series

$$\frac{4}{1\,.\,5} + \frac{4}{5\,.\,9} + \frac{4}{9\,.\,13} + \frac{4}{13\,.\,17} + \frac{4}{17\,.\,21} + \text{\&c., to infinity.}$$

Ans 1

CHAPTER IX.

CONTINUED FRACTIONS, EXPONENTIAL QUANTITIES, LOGARITHMS, AND FORMULAS FOR INTEREST.

248. Every expression of the form

$$\cfrac{1}{a + \cfrac{1}{b}}, \quad \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c}}}, \quad \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d}}}},$$

in which a, b, c, d, &c., are positive whole numbers, is called a *continued fraction.*

Hence, *a continued fraction has for its numerator the unit* 1, *and for its denominator a whole number, plus a fraction which has* 1 *for its numerator, and for its denominator a whole number plus a fraction, and so on.*

249. The resolution of an equation of the form

$$a^x = b,$$

gives rise to continued fractions. Suppose for example, $a = 8$, $b = 32$. We then have

$$8^x = 32,$$

in which $x > 1$, and $x < 2$. Make

$$x = 1 + \frac{1}{y},$$

in which $y > 1$, and the proposed equation becomes, after changing the members,

$$32 = 8^{1+\frac{1}{y}} = 8 \times 8^{\frac{1}{y}}, \quad \text{whence,}$$

$$8^{\frac{1}{y}} = 4, \quad \text{and consequently,} \quad 8 = 4^y.$$

It is plain, that the value of y lies between 1 and 2. Suppose

$$y = 1 + \frac{1}{z},$$

and we have, $\quad 8 = 4^{1+\frac{1}{z}} = 4 \times 4^{\frac{1}{z}};$

hence, $\quad 4^{\frac{1}{z}} = 2, \text{ and } 4 = 2^z, \text{ or } z = 2.$

But, $\quad y = 1 + \frac{1}{z} = 1 + \frac{1}{2} = \frac{3}{2};$

and $\quad x = 1 + \frac{1}{y} = 1 + \frac{1}{\frac{3}{2}} = 1 + \frac{2}{3} = \frac{5}{3};$

and this value will satisfy the proposed equation. For,

$$8^x = 8^{\frac{5}{3}} = \sqrt[3]{8^5} = \sqrt[3]{(2^3)^5} = \sqrt[3]{(2^5)^3} = 2^5 = 32.$$

250. If we apply a similar process to the equation

$$10^x = 200,$$

we shall find

$$x = 2 + \frac{1}{y}; \quad y = 3 + \frac{1}{z}; \quad z = 3 + \frac{1}{u}; \quad u = 3 + \frac{1}{t}.$$

Since 200 is not an exact power, x cannot be expressed either by a whole number or a fraction: hence, the value of x will be incommensurable, and the continued fraction will not terminate, but will be of the form

$$x = 2 + \frac{1}{y} = 2 + \cfrac{1}{3 + \cfrac{1}{z}} = 2 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{u + \&c.}}}$$

251. Common fractions may also be placed under the form of continued fractions.

Let us take, for example, the fraction $\frac{65}{149}$, and divide both its terms by the numerator 65, the value of the fraction will not be changed, and we shall have

$$\frac{65}{149} = \frac{1}{\frac{149}{65}},$$

or effecting the division, $\quad \frac{65}{149} = \cfrac{1}{2 + \cfrac{19}{65}}.$

Now, if we neglect the fractional part $\frac{19}{65}$ of the denominator, we shall obtain $\frac{1}{2}$ for the approximate value of the given fraction. But this value would be too large, since the *denominator* used was too *small*.

If, on the contrary, instead of neglecting the part $\frac{19}{65}$, we were to replace it by 1, the approximate value would be $\frac{1}{3}$, which would be too small, since the denominator 3 is too *large*. Hence,

$$\frac{65}{149} < \frac{1}{2} \quad \text{and} \quad \frac{65}{149} > \frac{1}{3},$$

therefore the value of the fraction is comprised between $\frac{1}{2}$ and $\frac{1}{3}$.

If we wish a nearer approximation, it is only necessary to operate on the fraction $\frac{19}{65}$ as we did on the given fraction $\frac{65}{149}$, and we obtain

$$\frac{19}{65} = \cfrac{1}{3 + \cfrac{8}{19}},$$

hence,

$$\frac{65}{149} = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{8}{19}}}.$$

If now, we neglect the part $\frac{8}{19}$, the denominator 3 will be less than the true denominator, and $\frac{1}{3}$ will be *larger* than the number which ought to be added to 2; hence, 1 divided by $2 + \frac{1}{3}$ will be *less* than the true value of the fraction; that is, if we stop at the first reduction and omit the fractional numbers, the result will be too great; if at the second, it will be too small, &c. Hence, generally, *if we stop at an odd reduction, and neglect the fractional part, the result will be too great; but if we stop at an even reduction, and neglect the fractional part, the result will be too small.*

Making two more reductions in the last example, we have

$$\frac{65}{149} = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \frac{1}{2}}}}}}$$

- - - - 1st reduction, too great;
- - 2d " too small;
- 3d " too great;
4th " too small;
5th " too great.

252. The separate fractions

$$\frac{1}{a}, \qquad \cfrac{1}{a + \cfrac{1}{b}}, \qquad \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c}}},$$

are called *approximating fractions*, because each affords, in succession, a nearer value of the given expression.

The fractions $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, &c., are called *integral fractions*.

When the expression can be exactly expressed by a vulgar fraction, as in the numerical examples already given, the integral fractions $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, &c., will terminate, and we shall obtain an expression for the exact value of the given fraction by taking them all.

We will now explain the manner in which any approximating fraction may be found from those which precede it.

1. $\frac{1}{a}$ - - - - - $= \frac{1}{a}$ 1st app. fraction.

2. $\cfrac{1}{a + \cfrac{1}{b}}$ - - - - $= \frac{b}{ab + 1}$ 2d app. fraction.

3. $\cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c}}}$ - - $= \frac{bc + 1}{(ab + 1)c + a}$ 3d app. fraction

By examining the third approximating fraction, we see, that its numerator is formed by multiplying the numerator of the preceding fraction by the denominator of the third integral fraction, and

adding to the product the numerator of the first approximating fraction: and that the denominator is formed by multiplying the denominator of the last fraction by the denominator of the third integral fraction, and adding to the product the denominator of the first approximating fraction.

We should infer, from analogy, that this law of formation is general. But to prove it rigorously, let $\frac{P}{P'}$, $\frac{Q}{Q'}$, $\frac{R}{R'}$, be any three approximating fractions for which the law is already established. Since c is the denominator of the last integral fraction, we have from what has already been proved,

$$\frac{R}{R'} = \frac{Qc + P}{Q'c + P'}.$$

Let us now add a new integral fraction $\frac{1}{d}$ to those already reduced, and suppose $\frac{S}{S'}$ to express the next approximating fraction. It is plain that $\frac{R}{R'}$ will become $\frac{S}{S'}$ by simply substituting for c, $c + \frac{1}{d}$: hence,

$$\frac{S}{S'} = \frac{Q\left(c + \frac{1}{d}\right) + P}{Q'\left(c + \frac{1}{d}\right) + P'} = \frac{(Qc + P)d + Q}{(Q'c + P')d + Q'} = \frac{Rd + Q}{R'd + Q'}.$$

Hence, we see that the fourth approximating fraction is deduced from the two immediately preceding it, in the same way that the third was reduced from the second and first; and as any fraction may be deduced from the two immediately preceding in a similar manner, we conclude that, *the numerator of the* n$^{\text{th}}$ *approximating fraction is formed by multiplying the numerator of the preceding fraction by the denominator of the* n$^{\text{th}}$ *integral fraction, and adding to the product the numerator of the* n − 2 *fraction; and the denominator is formed according to the same law, from the two preceding denominators.*

253. If we take the difference between any two of the consecutive approximating fractions, we shall find, after reducing them to a common denominator, that the difference of their numerators

will be equal to ± 1; and the denominator of this difference will be the product of the denominators of the fractions.

Taking, for example, the consecutive fractions $\frac{1}{a}$, and $\frac{b}{ab+1}$, we have

$$\frac{1}{a} - \frac{b}{ab+1} = \frac{ab+1-ab}{a(ab+1)} = \frac{+1}{a(ab+1)},$$

and $$\frac{b}{ab+1} - \frac{bc+1}{(ab+1)c+a} = \frac{-1}{(ab+1)[(ab+1)c+a]}.$$

To prove this property in a general manner, let

$$\frac{P}{P'}, \frac{Q}{Q'}, \frac{R}{R'},$$

be three consecutive approximating fractions. Then

$$\frac{P}{P'} - \frac{Q}{Q'} = \frac{PQ' - P'Q}{P'Q'};$$

and $$\frac{Q}{Q'} - \frac{R}{R'} = \frac{R'Q - RQ'}{Q'R'}.$$

But $R = Qc + P$ and $R' = Q'c + P'$ (Art. 252).

Substituting these values in the last equation, we have

$$\frac{Q}{Q'} - \frac{R}{R'} = \frac{(Q'c+P')Q - (Qc+P)Q'}{R'Q'};$$

or reducing

$$\frac{Q}{Q'} - \frac{R}{R'} = \frac{P'Q - PQ'}{R'Q'}$$

From which we see, that the numerator of the difference $\frac{P}{P'} - \frac{Q}{Q'}$ is equal, with a contrary sign, to that of the difference $\frac{Q}{Q'} - \frac{R}{R'}$. That is, *the difference between the numerators of any two consecutive approximating fractions, when reduced to a common denominator, is the same with a contrary sign, as that which exists between the last numerator and the numerator of the fraction immediately following.*

But we have already seen that the difference of the numerators of the 1st and 2d fractions is equal to $+1$; also that the difference between the numerators of the 2d and 3d fractions is equal to -1; hence, the difference between the numerators of the 3d and 4th is equal to $+1$; and so on for the following fractions.

Since the odd approximating fractions are all greater than the true value of the continued fraction, and the even ones all less (Art. 251), it follows, that when a fraction of an even order is subtracted from one of an odd order, the difference should have a plus sign; and on the contrary, it ought to have a minus sign, when one of an odd order is subtracted from one of an even.

254. It has already been shown (Art. 251), that each of the approximating fractions corresponding to the odd numbers, exceeds the true value of the continued fraction; while each of those corresponding to the even numbers, is less than it. Hence, the difference between any two consecutive fractions is greater than the difference between either of them and the true value of the continued fraction. Therefore, stopping at the n^{th} fraction, the result will be true to within 1 divided by the denominator of the n^{th} fraction, multiplied by the denominator of the fraction which follows. Thus, if Q' and R' are the denominators of consecutive fractions, and we stop at the fraction whose denominator is Q', the result will be true to within $\frac{1}{Q'R'}$. But since a, b, c, d, &c., are entire numbers, the denominator R' will be greater than Q', and we shall have

$$\frac{1}{Q'R'} < \frac{1}{Q'^2}.$$

hence, if the result be true to within $\frac{1}{Q'R'}$, it will certainly be true to within less than the larger quantity

$$\frac{1}{Q'^2};$$

that is, *the approximate result which is obtained, is true to within unity divided by the square of the denominator of the last approximating fraction that is employed.*

If we take the fraction $\frac{829}{347}$, we have

$$\frac{829}{347} = 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{19}}}}}.$$

Here, we have in the quotient the whole number 2, which may either be set aside, and added to the fractional part after its value shall have been found, or we may place 1 under it for a denominator, and treat it as an approximating fraction.

Resolution of the Equation $a^x = b$.

255. An equation of the form

$$a^x = b,$$

is called an *exponential equation*. The object in resolving this equation is, to find the exponent of the power to which it is necessary to raise a given number a, in order to produce another given number b.

Suppose it were required to resolve the equation

$$2^x = 64.$$

By raising 2 to its different powers, we find that $2^6 = 64$; hence, $x = 6$ will satisfy the conditions of the equation.

Again, let there be the equation

$$3^x = 243, \quad \text{in which} \quad x = 5.$$

In fact, so long as the second member b is a *perfect power* of the given number a, it may be obtained by raising a to its successive powers, commencing at the first.

Suppose it were required to resolve the equation

$$2^x = 6.$$

By making $x = 2$, and $x = 3$, we find

$$2^2 = 4 \quad \text{and} \quad 2^3 = 8;$$

from which we perceive that the value of x is comprised between 2 and 3.

Make then, $\quad x = 2 + \dfrac{1}{x'}, \quad$ in which $\quad x' > 1.$

Substituting this value in the given equation, it becomes,

$$2^{2+\frac{1}{x'}} = 6, \quad \text{or} \quad 2^2 \times 2^{\frac{1}{x'}} = 6; \quad \text{hence,}$$

$$2^{\frac{1}{x'}} = \frac{6}{4} = \frac{3}{2};$$

and by changing the terms and raising both members to the x' power,

$$\left(\frac{3}{2}\right)^{x'} = 2$$

To determine x', make successively $x' = 1$ and 2; we find

$$\left(\frac{3}{2}\right)^1 = \frac{3}{2} < 2; \text{ and } \left(\frac{3}{2}\right)^2 = \frac{9}{4} > 2;$$

therefore, x' is comprised between 1 and 2.

Make, $\qquad x' = 1 + \frac{1}{x''}$, in which $x'' > 1$.

By substituting this value in the equation $\left(\frac{3}{2}\right)^{x'} = 2$,

$$\left(\frac{3}{2}\right)^{1+\frac{1}{x''}} = 2; \text{ hence, } \frac{3}{2} \times \left(\frac{3}{2}\right)^{\frac{1}{x''}} = 2,$$

and consequently, $\qquad \left(\frac{4}{3}\right)^{x''} = \frac{3}{2}.$

The hypothesis $\quad x'' = 1, \quad$ gives $\frac{4}{3} < \frac{3}{2}$;

and of $\quad x'' = 2, \quad$ gives $\frac{16}{9} > \frac{3}{2}$;

therefore, x'' is comprised between 1 and 2.

Let $\qquad x'' = 1 + \frac{1}{x'''}$; then,

$$\left(\frac{4}{3}\right)^{1+\frac{1}{x'''}} = \frac{3}{2}; \text{ hence, } \frac{4}{3} \times \left(\frac{4}{3}\right)^{\frac{1}{x'''}} = \frac{3}{2};$$

whence, $\qquad \left(\frac{9}{8}\right)^{x'''} = \frac{4}{3}.$

If we make $x''' = 2$, we have

$$\left(\frac{9}{8}\right)^2 = \frac{81}{64} < \frac{4}{3},$$

and if we make $x''' = 3$, we have

$$\left(\frac{9}{8}\right)^3 = \frac{729}{512} > \frac{4}{3};$$

therefore, x''' is comprised between 2 and 3.

Make $\qquad x''' = 2 + \frac{1}{x^{\text{IV}}}$, and we have

$$\left(\frac{9}{8}\right)^{2+\frac{1}{x^{\text{IV}}}} = \frac{4}{3}; \text{ hence, } \frac{81}{64}\left(\frac{9}{8}\right)^{\frac{1}{x^{\text{IV}}}} = \frac{4}{3};$$

and consequently, $\qquad \left(\frac{256}{243}\right)^{x^{\text{IV}}} = \frac{9}{8}.$

Operating upon this exponential equation in the same manner as upon the preceding equations, we shall find two entire numbers, k and $k+1$, between which x^{IV} will be comprised.

Making

$$x^{\text{IV}} = k + \frac{1}{x^{\text{V}}},$$

and x^{V} can be determined in the same manner as x^{IV}, and so on.

Making the necessary substitutions in the equations

$$x = 2 + \frac{1}{x'}, \quad x' = 1 + \frac{1}{x''}, \quad x'' = 1 + \frac{1}{x'''}, \quad x''' = 2 + \frac{1}{x^{\text{IV}}} \; \ldots,$$

we obtain the value of x under the form of a continued fraction

$$x = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{x^{\text{IV}}}}}}.$$

Hence, we find the first three approximating fractions to be

$$\frac{1}{1}, \quad \frac{1}{2}, \quad \frac{3}{5},$$

and the fourth is equal to

$$\frac{3 \times 2 + 1}{5 \times 2 + 2} = \frac{7}{12} \text{ (Art. 252)},$$

which is the true value of the fractional part of x to within

$$\frac{1}{(12)^2} \text{ or } \frac{1}{144} \text{ (Art. 254)}.$$

Therefore,

$$x = 2 + \frac{7}{12} = \frac{31}{12} = 2.58333 + \quad \text{to within } \frac{1}{144},$$

and if a greater degree of exactness is required, we must take a greater number of integral fractions.

EXAMPLES.

$3^x = 15$	. . . $x =$	2.46	to within	0.01.
$10^x = 3$	. . . $x =$	0.477	"	0.001.
$5^x = \frac{2}{3}$	. . . $x =$	-0.25	"	0.01

Theory of Logarithms.

256. If we suppose a to preserve the same value in the equation

$$a^x = N,$$

and N become, in succession, every possible positive number, it is plain that x will undergo changes corresponding to those made in N. By the method explained in the last Article, we can determine, for each value of N, the corresponding value of x, either exactly or approximatively.

Any number, except 1, may be taken for the invariable number a; but when once chosen, it is supposed to remain the same for the formation of one entire series of numbers.

The exponent x of a, corresponding to any value of N, is called the *logarithm* of that number; and the invariable number a is called the *base* of that system of logarithms. Hence,

The logarithm of a number, is the exponent of the power to which it is necessary to raise an invariable number, called the base of the system, in order to produce the number.

The general properties of logarithms are independent of any particular base. The use that may be made of them in numerical calculations, supposes the construction of a table, containing all the numbers in one column, and the *logarithms* of these numbers in another, calculated from a given *base*. Now, in calculating this table, it is necessary, in considering the equation

$$a^x = N,$$

to make N pass through all possible states of value, and to determine the value of x corresponding to each of the values of N, which may be done by the method of Art. 255.

257. The base of the common system of logarithms, or as they are sometimes called, Briggs' logarithms, from their inventor, is the nnmber 10. If we designate the logarithm of any number by log. or l, we shall have

$$\begin{array}{rlrl}
(10)^0 = & 1; & \text{hence,} & \log.\ 1 = 0; \\
(10)^1 = & 10; & \text{hence,} & \log.\ 10 = 1; \\
(10)^2 = & 100; & \text{hence,} & \log.\ 100 = 2; \\
(10)^3 = & 1000; & \text{hence,} & \log.\ 1000 = 3; \\
(10)^4 = & 10000; & \text{hence,} & \log.\ 10000 = 4. \\
& \&c., & & \&c.
\end{array}$$

Hence, in the common system, the logarithm of any number between 1 and 10, is >0 and <1. The logarithm of any number between 10 and 100, is >1 and <2; the logarithm of any number between 100 and 1000, is >2 and <3; and so on.

Hence, the logarithm of any number expressed by two figures, and which is not a perfect power of the base of the system, will be equal to a whole number plus an approximating fraction, the approximate value of which fraction is generally expressed decimally.

The integral part of a logarithm, is called the *index* or *characteristic of the logarithm.*

By examining the several powers of 10, we see, that if a number is expressed by a single figure, the characteristic of its logarithm will be 0; if it is expressed by two figures, the characteristic of its logarithm will be 1; if it is expressed by three figures, the characteristic will be 2; and if it is expressed by n places of figures, the characteristic will be $n-1$ units.

The following table shows the logarithms of the numbers, from 1 to 100.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0.000000	26	1.414973	51	1.707570	76	1.880814
2	0.301030	27	1.431364	52	1.716003	77	1.886491
3	0.477121	28	1.447158	53	1.724276	78	1.892095
4	0.602060	29	1.462398	54	1.732394	79	1.897627
5	0.698970	30	1.477121	55	1.740363	80	1.903090
6	0.778151	31	1.491362	56	1.748188	81	1.908485
7	0.845098	32	1.505150	57	1.755875	82	1.913814
8	0.903090	33	1.518514	58	1.763428	83	1.919078
9	0.954243	34	1.531479	59	1.770852	84	1.924279
10	1.000000	35	1.544068	60	1.778151	85	1.929419
11	1.041393	36	1.556303	61	1.785330	86	1.934498
12	1.079181	37	1.568202	62	1.792392	87	1.939519
13	1.113943	38	1.579784	63	1.799341	88	1.944483
14	1.146128	39	1.591065	64	1.806180	89	1.949390
15	1.176091	40	1.602060	65	1.812913	90	1.954243
16	1.204120	41	1.612784	66	1.819544	91	1.959041
17	1.230449	42	1.623249	67	1.826075	92	1.963788
18	1.255273	43	1.633468	68	1.832509	93	1.968483
19	1.278754	44	1.643453	69	1.838849	94	1.973128
20	1.301030	45	1.653213	70	1.845098	95	1.977724
21	1.322219	46	1.662758	71	1.851258	96	1.982271
22	1.342423	47	1.672098	72	1.857333	97	1.986772
23	1.361728	48	1.681241	73	1.863323	98	1.991226
24	1.380211	49	1.690196	74	1.869232	99	1.995635
25	1.397940	50	1.698970	75	1.875061	100	2.000000

The characteristic being always one less than the number of places of figures in the number, is not written down in the table of logarithms for numbers which exceed 100. Thus, in searching for the logarithm of 2970, we should find in the table opposite 2970, the decimal part .472756. But since the number is expressed by four figures, the characteristic of the logarithm is 3. Hence,

$$\log. 2970 = 3.472756,$$

and by the definition of a logarithm, the equation

$$a^x = N, \quad \text{gives}$$

$$10^{3.472756} = 2970.$$

Multiplication and Division by Logarithms.

258. Let a be the base of a system of logarithms, and suppose the table to be calculated. Let it be required to multiply together a series of numbers by means of their logarithms. Denote the numbers by N, N', N'', N''', &c., and their corresponding logarithms by x, x', x'', x''', &c. Then, by definition (Art. 257), we have

$$a^x = N', \quad a^{x'} = N', \quad a^{x''} = N'', \quad a^{x'''} = N''' \ . \ . \ . \ \&c.$$

Multiplying these equations together, member by member, and applying the rule for the exponents, we have

$$a^{x+x'+x''+x'''} \ . \ . \ . = N \times N' \times N'' \times N''' \ . \ . \ . \ .$$

But since a is the base of the system, we have

$$x + x' + x'' + x''' \ . \ . \ . = \log. (N, N', N'', N''' \ . \ . \ . \ . \ );$$

that is, *the sum of the logarithms of any number of factors, is equal to the logarithm of the product of those factors.*

259. Suppose it were required to divide one number by another. Let N and N' denote the numbers, and x and x' their logarithms. We have the equations

$$a^x = N \quad \text{and} \quad a^{x'} = N';$$

hence, by division

$$\frac{a^x}{a^{x'}} = a^{x-x'} = \frac{N}{N'};$$

or

$$x - x' = \log. N - \log. N' = \log. \left(\frac{N}{N'}\right),$$

that is, *the difference between the logarithm of the dividend and the logarithm of the divisor, is equal to the logarithm of the quotient.*

Consequences of these Properties.

A multiplication can be performed by taking the logarithms of the two factors from the tables, and *adding* them together; this will give the logarithm of the product. Then finding this new logarithm in the tables, and taking the number which corresponds to it, we shall obtain the required product. Therefore, *by a simple addition, we find the product arising from a multiplication.*

In like manner, when one number is to be divided by another, subtract the logarithm of the divisor from that of the dividend, then find the number corresponding to this difference; this will be the required quotient. Therefore, *by a simple subtraction, we obtain the quotient arising from a division.*

Formation of Powers and Extraction of Roots.

260. Let it be required to raise a number N to any power denoted by $\frac{m}{n}$. If a denotes the base of the system, and x the logarithm of N, we shall have

$$a^x = N, \quad \text{or} \quad N = a^x;$$

whence, by raising both members to the power $\frac{m}{n}$,

$$N^{\frac{m}{n}} = a^{\frac{m}{n}x}.$$

Therefore, $$\log.\left(N^{\frac{m}{n}}\right) = \frac{m}{n}\,.\,x = \frac{m}{n}\,.\log. N.$$

If we make $n = 1$; there will result,

$$m\,.\log. N = \log. N^m;$$

an equation which is susceptible of the following enunciation:

If the logarithm of any number be multiplied by the exponent of the power to which the number is to be raised, the product will be equal to the logarithm of that power.

261. Suppose, in the first equation, $m = 1$; there will result,

$$\frac{1}{n}\log. N = \log. N^{\frac{1}{n}} = \log. \sqrt[n]{N}; \quad \text{that is,}$$

The logarithm of any root of a number is obtained, by dividing the logarithm of the number by the index of the root.

Consequences.

To form any power of a number, take the logarithm of this number from the tables, multiply it by the exponent of the power; then the number corresponding to this product will be the required power.

In like manner, to extract the root of a number, divide the logarithm of the proposed number by the index of the root; then the number corresponding to the quotient will be the required root. Therefore, *by a simple multiplication, we can raise a quantity to a power, and extract its root by a simple division.*

262. If we make the different exponents of 10 negative, the powers corresponding thereto will be decimal fractions. Thus,

$$(10)^{-1} = \frac{1}{10} = 0.1; \quad \text{hence,} \quad \log. 0.1 = -1;$$
$$(10)^{-2} = \frac{1}{100} = 0.01; \quad \text{hence,} \quad \log. 0.01 = -2;$$
$$(10)^{-3} = \frac{1}{1000} = 0.001; \quad \text{hence,} \quad \log. 0.001 = -3;$$
$$(10)^{-4} = \frac{1}{10000} = 0.0001; \quad \text{hence,} \quad \log. 0.0001 = -4.$$
&c., &c., &c.

The logarithm of any fraction between one and one tenth, and as four tenths, for example, may be expressed thus,

$$\log. \left(\frac{4}{10}\right) = \log. \left(\frac{1}{10} \times 4\right) = \log. \frac{1}{10} + \log. 4 = -1 + \log. 4.$$

For the fractions between one hundredth and one tenth, as six hundredths, for example, we have

$$\log. \left(\frac{6}{100}\right) = \log. \left(\frac{1}{100} \times 6\right) = -2 + \log. 6.$$

For the fractions between one thousandth and one hundredth, as eight thousandths, for example, we have

$$\log. \left(\frac{8}{1000}\right) = \log. \left(\frac{1}{1000} \times 8\right) = -3 + \log. 8.$$

Now, instead of performing the subtractions indicated above, we unite the decimal part of the logarithm to the negative characteristic. Thus,

$$\log. 0.4 = -1 + \log. 4 = -1.602060;$$
$$\log. 0.06 = -2 + \log. 6 = -2.778151;$$
$$\log. 0.008 = -3 + \log. 8 = -3.903090.$$

Adopting this method of writing the logarithms, we see that the logarithm of a decimal fraction may be found from the tables, by *uniting to the logarithm of its numerator, regarded as a whole number, a negative characteristic greater by unity than the number of ciphers between the decimal point and the first significant figure.*

To demonstrate this in a general manner, let a denote the numerator of a decimal fraction, and b its denominator. From the nature of decimals, we shall have

$$b = (10)^m,$$

in which m will denote the number of ciphers in the denominator. Hence

$$\log.\frac{a}{b} = \log.\left(\frac{a}{(10)^m}\right) = \log.a - m\log.10 = \log.a - m.$$

Or, in other words, *the logarithm of a whole number will become the logarithm of a corresponding decimal, by adding to it a negative characteristic containing as many units as there are ciphers in the denominator of the decimal fraction.*

Hence, the table of logarithms whose base is 10, will give the logarithms of all decimals, as well as of the integral numbers.

GENERAL EXAMPLES.

1. What is the square of 7?

Log. 7 - - - - -	-	$= 0.845098$
Exponent of the power	-	$= \qquad 2$
Number corresponding, 49	-	$1.690196.$

2. What is the 6th power of 2?

Log. 2 - - - - -	-	$= 0.301030$
Exponent of the power -	-	$= \qquad 6$
Number corresponding, 64	-	$1.806180.$

3. What is the cube root of 64?

Log. 64 - - - -	-	$= 1.806180$
Then, - - - -	-	$3)\ 1.806180$
Number corresponding, 4	-	$0.603060.$

4. What is the 4th root of 81? *Ans.* 3.

5. What is the 5th root of 32? *Ans.* 2.

6. Log. $(a \,.\, b \,.\, c \,.\, d \ldots\ldots) = \log. a + \log. b + \log. c \ldots\ldots$

7. Log. $\left(\dfrac{abc}{de}\right) = \log. a + \log. b + \log. c - \log. d - \log. e.$

8. Log. $(a^m \,.\, b^n \,.\, c^p \ldots\ldots) = m \log. a + n \log. b + p \log. c + \ldots\ldots$

9. Log. $(a^2 - x^2) = \log. (a + x) + \log. (a - x).$

10. Log. $\sqrt{(a^2 - x^2)} = \frac{1}{2} \log. (a + x) + \frac{1}{2} \log. (a - x).$

11. Log. $(a^3 \times \sqrt[4]{a^3}) = 3\frac{3}{4} \log. a.$

263. Let us resume the general equation

$$a^x = N,$$

and suppose a to be the base of a system of logarithms. Then,

1st, we have $a^1 = N = a$, whence, $\log. a = 1$;

2d " $a^0 = 1$, whence, $\log. 1 = 0$;

that is, *whatever be the base of the system, its logarithm taken in that system, is equal to* 1, *and the logarithm of* 1 *is equal to* 0.

264. Let us suppose, in the equation

$$a^x = N,$$

that $$a > 1.$$

Then, if we make $N = 1$, we shall have

$$a^0 = 1.$$

If we make $N < 1$, we must have

$$a^{-x} = N, \quad \text{or} \quad \frac{1}{a^x} = N < 1.$$

If now, N diminishes, x will increase, and when N becomes 0, we have

$$a^{-x} = \frac{1}{a^x} = 0, \quad \text{or} \quad a^x = \infty \text{ (Art. 112)};$$

but no finite power of a is infinite, hence $x = \infty$: and therefore, *the logarithm of* 0 *in a system of which the base is greater than unity, is an infinite number and negative.*

265. Again, take the equation

$$a^x = N,$$

and suppose the base $a < 1$. Then making, as before,

$N = 1$, we have $a^0 = 1$.

If we make N less than 1, we shall have

$$a^x = N < 1.$$

Now, if we diminish N, x will increase; for, since $a < 1$, its powers will diminish as the exponent x increases, and when $N = 0$, x must be infinite, for no finite power of a fraction can be 0. Hence, *the logarithm of* 0 *in a system of which the base is less than unity, is an infinite number and positive.*

Logarithmic and Exponential Series.

266. The method of resolving the equation

$$a^x = b,$$

explained in Art. 255, gives an idea of the construction of logarithmic tables; but this method is laborious when it is necessary to approximate very near the value of x. Analysts have discovered much more expeditious methods for constructing new tables, or for verifying those already calculated. These methods consist in the development of logarithms into series.

Taking again the equation

$$a^x = y,$$

it is proposed to develop the logarithm of y into a series involving the powers of y, and co-efficients independent of y.

It is evident, that the same number y will have a different logarithm in different systems, that is, for different values of the base a; hence, the log. y, will depend for its value, 1st, on the value of y; and 2dly, on a, the base of the system of logarithms. Hence, the development must contain y, or some quantity dependent on it, and some quantity dependent on the base a.

To find the form of this development, we will assume

$$\log. y = A + By + Cy^2 + Dy^3 +, \&c.,$$

in which A, B, C, &c., are independent of y, and dependent on the base a.

Now, if we make $y = 0$, the log. y becomes infinite, and is either negative or positive, according as the base a is greater or less than unity (Arts. 264 & 265). But the second member under this supposition, reduces to A, a finite number: hence, the development cannot be made under that form

Again, assume

$$\log. y = Ay + By^2 + Cy^3 + Dy^4 +, \text{ \&c.}$$

If we make $y = 0$, we have

$$\log. y = \pm \infty, \text{ that is, } \pm \infty = 0,$$

which is absurd, and hence the development cannot be made under the last form. Hence we conclude that, *the logarithm of a number cannot be developed in the powers of that number.*

Let us place, in the first member, $1 + y$ for y, and we have

$$\log. (1 + y) = Ay + By^2 + Cy^3 + Dy^4 + \text{\&c.} \ldots (1),$$

making $y = 0$, the equation is reduced to $\log. 1 = 0$, which does not present any absurdity.

In order to determine the co-efficients A, B, C, . . . we shall follow the process of Art. 243. Since equation (1) is true for any value of y, it will be true if we substitute z for y, and we may write

$$\log. (1 + z) = Az + Bz^2 + Cz^3 + Dz^4 + \ldots (2).$$

Subtracting equation (2) from (1), we obtain

$$\log. (1+y) - \log. (1+z) = A(y-z) + B(y^2-z^2) + C(y^3-z^3) + \ldots (3).$$

The second member of this equation is divisible by $y - z$. Let us see, if we can by any artifice, put the first under such a form that it shall also be divisible by $y - z$. We have,

$$\log. (1 + y) - \log. (1 + z) = \log. \left(\frac{1 + y}{1 + z}\right) = \log. \left(1 + \frac{y - z}{1 + z}\right).$$

But since $\dfrac{y - z}{1 + z}$ can be regarded as a single number u, we can develop $\log. (1 + u)$, or $\log. \left(1 + \dfrac{y - z}{1 + z}\right)$, in the same manner as $\log. (1 + y)$, which gives

$$\log. \left(1 + \frac{y - z}{1 + z}\right) = A\frac{y - z}{1 + z} + B\left(\frac{y - z}{1 + z}\right)^2 + C\left(\frac{y - z}{1 + z}\right)^3 + \ldots$$

Substituting this development for

$$\log. (1 + y) - \log. (1 + z),$$

in the equation (3), and dividing both members by $y - z$, it becomes

$$A \cdot \frac{1}{1 + z} + B\frac{y - z}{(1 + z)^2} + C\frac{(y - z)^2}{(1 + z)^3} + \ldots$$
$$= A + B(y + z) + C(y^2 + yz + z^2) + \ldots$$

Since this equation, like the preceding, is true for all values of y and z, make $y = z$, and there will result

$$\frac{A}{1+y} = A + 2By + 3Cy^2 + 4Dy^3 + 5Ey^4 + \ldots$$

whence, by making the terms entire, and transposing,

$$0 = \begin{cases} +A + 2B \\ -A + \;\; A \end{cases} \left| \begin{matrix} y + 3C \\ \;\;\; + 2B \end{matrix} \right| \begin{matrix} y^2 + 4D \\ \;\;\; + 3C \end{matrix} \left| \begin{matrix} y^3 + 5E \\ \;\;\; + 4D \end{matrix} \right| y^4 + \ldots$$

Placing the co-efficients of the different powers of y equal to zero, we obtain the series of equations

$$A - A = 0, \quad 2B + A = 0, \quad 3C + 2B = 0, \quad 4D + 3C = 0 \ldots;$$

whence,

$$A = A, \quad B = -\frac{A}{2}, \quad C = -\frac{2B}{3} = +\frac{A}{3}, \quad D = -\frac{3C}{4} = -\frac{A}{4} \ldots$$

The law of the series is evident; the co-efficient of the n^{th} term is equal to $\mp \frac{A}{n}$, according as n is even or odd: hence, we obtain for the development,

$$\log. (1 + y) = Ay - \frac{A}{2}y^2 + \frac{A}{3}y^3 - \frac{A}{4}y^4 \ldots$$

$$= A\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} \ldots\right) \quad (4).$$

Hence, although the logarithm of a number cannot be developed in the powers of that number, yet *it may be developed in the powers of a number less by unity.*

By the above method of development, the co-efficients B, C, D, E, &c., have all been determined in functions of A; but A remains entirely undetermined. This indeed should be so, since A depends for its value on the base of the system, to which any value may be assigned.

Denote by x' that part of the second member of equation (4) which involves y, and suppose a to be the base of the system in which the log. $(1 + y)$ is taken, and we have

$$a^{Ax'} = 1 + y, \quad \text{or} \quad Ax' = \log. (1 + y).$$

But the log. $(1 + y)$ depends for its value on two things: viz., on the number of units in y, and on the base of the system in which the logarithm is taken. The series denoted by x' is ex-

pressed in y, and hence depends for its value on y alone. But A being independent of y, its value must depend on the base of the system; and hence,

The expression for the logarithm of any number is composed of two factors, one dependent on the number, and the other on the base of the system in which the logarithm is taken. The factor which depends on the base, is called the *modulus* of the system of logarithms.

267. If we take the logarithm of $1 + y$ in a new system, and denote it by $l'.(1 + y)$, we shall have

$$l'.(1+y) = A'\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \&c.\right) \quad (5),$$

in which A' is the modulus of the new system.

If we suppose y to have the same value as in equation (4), we shall have

$$l'.(1+y) : l.(1+y) :: A' : A;$$

for, since the series in the second members are the same, they may be omitted. Therefore,

The logarithms of the same number, taken in two different systems, are to each other as the moduli of those systems.

268. Having shown that the modulus and base of a system of logarithms are mutually dependent on each other, it follows, that if a value be assigned to one of them, the corresponding value of the other must be determined from it.

If then, we make the modulus

$$A' = 1,$$

the base of the system will assume a fixed value. The system of logarithms resulting from such a modulus, and such a base, is called the *Naperian System.* This was the first system known, and was invented by Baron Napier, a Scotch mathematician.

With this modification, the proportion above becomes

$$l'.(1+y) : l.(1+y) :: 1 : A,$$

and $$A \,.\, l'.(1+y) = l.(1+y).$$

Hence we see that,

The Naperian logarithm of any number, multiplied by the modulus of another system, will give the logarithm of the same number in that system.

The modulus of the Naperian System being unity, it is found most convenient to compare all other systems with the Naperian; and hence, the modulus of any system may be defined to be,

The number by which it is necessary to multiply the Naperian logarithm in order to obtain the logarithm of the same number in the other system.

269. Again, $A \times l'.(1+y) = l.(1+y)$ gives

$$l'.(1+y) = \frac{l.(1+y)}{A}.$$

That is, *the logarithm of any number divided by the modulus of its system, is equal to the Naperian logarithm of the same number.*

270. If we take the Naperian logarithm and make $y = 1$, equation (5) becomes

$$l'.2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots$$

a series which does not converge rapidly, and in which it would be necessary to take a great number of terms to obtain a near approximation. In general, this series will not serve for determining the logarithms of entire numbers, since for every number greater than 2 we should obtain a series in which the terms would go on increasing continually.

The following are the principal transformations for converting the above series into converging series, for the purpose of obtaining the logarithms of entire numbers, which are the only logarithms placed in the tables.

First Transformation.

Taking the Naperian logarithm in equation (5), making $y = \frac{1}{z}$, and observing that

$$l'.\left(1 + \frac{1}{z}\right) = l'.(1+z) - l'.z, \quad \text{it becomes}$$

$$l'.(1+z) - l'.z = \frac{1}{z} - \frac{1}{2z^2} + \frac{1}{3z^3} - \frac{1}{4z^4} + \&c. \quad (6).$$

This series becomes more converging as z increases; besides, the first member of the equation expresses the difference between the logarithms of two consecutive numbers.

Making $z = 1, 2, 3, 4, 5$, &c., in succession, we have

$$l'.2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \&c.$$

$$l'.3 - l'.2 = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \ldots$$

$$l'.4 - l'.3 = \frac{1}{3} - \frac{1}{18} + \frac{1}{81} - \frac{1}{324} + \ldots$$

$$l'.5 - l'.4 = \frac{1}{4} - \frac{1}{32} + \frac{1}{192} - \frac{1}{1024}.$$

The first series will give the logarithm of 2; the second series will give the logarithm of 3 by means of the logarithm of 2; the third, the logarithm of 4, in functions of the logarithm of 3 . . . &c. The degree of approximation can be estimated, since the series are composed of terms alternately positive and negative (Art. 241).

Second Transformation.

A much more converging series is obtained in the following manner. In the series

$$l'.(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

substitute $-x$ for x, and it becomes

$$l'.(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$$

Subtracting the second series from the first, observing that

$$l'.(1 + x) - l'.(1 - x) = l'.\left(\frac{1 + x}{1 - x}\right), \text{ we obtain}$$

$$l'.\left(\frac{1 + x}{1 - x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \ldots\right).$$

This series will not converge very rapidly unless x is a very small fraction, in which case, $\frac{1 + x}{1 - x}$ will be greater than unity, but will differ very little from it.

Make $\frac{1 + x}{1 - x} = 1 + \frac{1}{z}$, z being an entire number. We have

$$(1 + x)z = (1 - x)(z + 1); \quad \text{whence,} \quad x = \frac{1}{2z + 1}.$$

Hence, the preceding series becomes $l'.\left(1+\frac{1}{z}\right)$, or

$$l'.(z+1)-l'.z=2\left(\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+\ldots\right).$$

This series gives the difference between the logarithms of two consecutive numbers, and converges more rapidly than series (6)

Making successively, $z=1, 2, 3, 4, 5 \ldots$, we find

$$l'.2=2\left(\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\frac{1}{7.3^7}+\ldots\right),$$

$$l'.3-l'.2=2\left(\frac{1}{5}+\frac{1}{3.5^3}+\frac{1}{5.5^5}+\frac{1}{7.5^7}+\ldots\right),$$

$$l'.4-l'.3=2\left(\frac{1}{7}+\frac{1}{3.7^3}+\frac{1}{5.7^5}+\frac{1}{7.7^7}+\ldots\right),$$

$$l'.5-l'.4=2\left(\frac{1}{9}+\frac{1}{3.9^3}+\frac{1}{5.9^5}+\frac{1}{7.9^7}\ldots\ldots\right).$$

. . . .

. . . .

Let $z=100$; there will result

$$l'.101=l'.100+2\left(\frac{1}{201}+\frac{1}{3(201)^3}+\frac{1}{5(201)^5}+\ldots\right);$$

whence we see, that knowing the logarithm of 100, the first term of the series is sufficient for obtaining that of 101 to seven places of decimals.

There are formulas more converging than the above, from which we may obtain a series of logarithms in functions of others already known, but the preceding are sufficient to give an idea of the facility with which tables may be constructed. We may now suppose the Naperian logarithms of all numbers to be known.

The Naperian logarithm of 10 may be deduced from the first and fourth of the above equations, by simply adding the logarithm of 2 to that of 5 (Art. 258). This number has been calculated with great exactness, and is 2.302585093.

271. We have already observed that the base of the common system of logarithms is 10 (Art. 257). We will now find its modulus. We have,

$$l'.(1+y) : l.(1+y) :: 1 : A \quad \text{(Art. 267).}$$

If we make $y = 9$, we shall have

$$l'.10 : l.10 :: 1 : A.$$

But the $l'.10 = 2.302585093$, and $l.10 = 1$ (Art. 257);

hence, $A = \dfrac{1}{2.302585093} = 0.434294482 =$ the modulus of the common system.

If now, we multiply the Naperian logarithms before found, by this modulus, we shall obtain a table of common logarithms (Art. 268).

All that now remains to be done is to find the base of the Naperian system. If we designate that base by e, we shall have (Art. 267),

$$l'.e : l.e :: 1 : 0.434294482.$$

But $l'e = 1$ (Art. 263): hence,

$$l. : l.e :: 1 : 0.434294482,$$

hence, $$l.e = 0.434294482.$$

But as we have already explained the method of calculating the common tables, we may use them to find the number whose logarithm is 0.434294482, which we shall find to be 2.718281828: hence

$$e = 2.718281828.$$

We see from the last equation but one that, *the modulus of the common system is equal to the common logarithm of the Naperian base.*

Of Interpolation.

272. A table of logarithms is a tabulated series of numbers, showing the value of x in the equation

$$a^x = N,$$

corresponding to all the integral values of N, between 1 and some higher number which marks the limit of the table. It has already been remarked that in the system in common use, the value of the base a, is 10.

And generally, *any mathematical table consists of a series of values of some letter in an algebraic expression, corresponding to equi-distant values of the function on which it depends.*

The principle of interpolation, which is of great value in practical science, has for its object to find from the tabulated numbers

which are given, other similar numbers which shall correspond to intermediate values of the function. For example, suppose p, q, r, s, &c., to be a series of tabulated numbers corresponding to, and written opposite the functions a, $a+b$, $a+2b$, $a+3b$, &c., and it were required to find the tabulated number corresponding to the function $a+2\frac{1}{2}b$. This is a question of interpolation, and is resolved by taking the successive differences of the tabulated numbers, thus:

Functions.	Differences of tabulated numbers.			
-	-	-	-	-
-	-	-	-	-
a	p	-	-	-
		dp	-	-
$a+b$	q		d^2p	-
		dq		d^3p
$a+2b$	r		d^2q	
		dr		d^3q
$a+3b$	s		d^2r	-
		ds	-	-
$a+4b$	t		-	-
-	-	-	-	-
-	-	-	-	-

in which $dp = q - p$, $\quad dq = r - q$, $\quad dr = s - r$, &c;
and $d^2p = dq - dp$, $\quad d^2q = dr - dq$, $\quad d^2r = ds - dr$, &c.;
also, $d^3p = d^2q - d^2p$, $\quad d^3q = d^2r - d^2q$, &c.
&c., &c.

From the above equations, we have

$$q = p + dp, \quad r = q + dq = p + dp + dq = p + 2dp + d^2p;$$

and by a similar process, we have

$$s = p + 3dp + 3d^2p + d^3p,$$
$$t = p + 4dp + 6d^2p + 4d^3p + d^4p,$$
&c., &c.,

in which notation it should be observed, that d^2, d^3, &c., denote the second, third, &c. differences of the successive tabulated numbers.

It is plain, that the above law from which the numerical co-efficients for any term may be derived, is similar to that for the co-efficients of a binomial: hence, if T denote the $n+1$ term of the tabulated numbers, reckoning from p inclusive, we shall have

$$T = p + ndp + \frac{n(n-1)}{1\,.\,2}d^2p + \frac{n\,.\,(n-1)(n-2)}{1\,.\,2\,.\,3}d^3p + \text{\&c.}$$

Let it be required to find the tabulated number corresponding to $a + 3b$. We then have, $n = 3$: hence,

$$T = p + 3dp + 3d^2p + d^3p,$$

the same value as that found above for r.

Next, let it be required to find the tabulated value answering to the functions $a + \frac{3}{4}b$. Then, $n = \frac{3}{4}$, and if we know the tabulated number p, and the successive differences d, d^2, &c., the approximate value of T can easily be found.

It is plain from the series that the interpolated values are but approximations, since no order of difference can reduce to zero; and hence, the series will contain an infinite number of terms. Generally, however, the tabulated values are themselves but approximations, and the successive differences decrease so rapidly in value, that the series becomes very converging.

Let us suppose for example, that we have the logarithms of 12, 13, 14, 15, &c., and that it is required to find the logarithm of 12 and a half. Then,

12	1.079181		
		$dp = 0.034762$	
13	1.113943		$d^2p = -0.002577$
		$dq = 0.032185$	
14	1.146128		$d^2q = -0.002222$
		$dr = 0.029963$	
15	1.176091		

also, $d^3p = d^2q - d^2p = +0.000355$.

Making $n = \dfrac{1}{2}$, and stopping at the term involving the third difference, we have

$$T = p + \frac{1}{2}dp + \frac{1}{2} \times -\frac{1}{4}d^2p + \frac{1}{2} \times -\frac{1}{4} \times -\frac{1}{2}d^3p + \&c.$$

$$\begin{aligned}
&+ \ p \quad - \ - \ - \ - \ - \ - &= 1.079181 \\
&+ \ \tfrac{1}{2}dp \quad - \ - \ - \ - \ - \ - &= 0.017381 \\
&+ \ \tfrac{1}{8}d^2p \quad - \ - \ - \ - \ - \ - &= 0.000322 \\
&+ \ \tfrac{1}{16}d^3p \quad - \ - \ - \ - \ - \ - &= \underline{0.000022} \\
&T = \log. 12\tfrac{1}{2} &= \underline{1.096906}.
\end{aligned}$$

INTEREST.

273. The solution of all questions relating to interest, may be greatly simplified by employing the algebraic formulas.

In treating of this subject, we shall employ the following notation:

Let $p =$ the amount bearing interest, called the *principal;*

$r =$ the part of \$1, which expresses its interest for one year, called the *rate per cent.*;

$t =$ the time that p draws interest;

$i =$ the interest of p dollars for t years;

$S = p +$ the interest which accrues in the time t, which is called the *amount*.

Simple Interest.

To find the interest of a sum p *for* t *years, at the rate* r, *and the amount then due.*

Since r denotes the part of a dollar which expresses its interest for a single year, the interest of p dollars for the same time will be expressed by pr; and for t years it will be t times as much: hence,

$$i = ptr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1);$$

and for the amount due,

$$S = p + ptr = p(1 + tr) \quad . \quad . \quad (2).$$

EXAMPLES.

1. What is the interest, and what the amount of \$365 for three years and a half, at the rate of 4 per cent. per annum. Here,

$$p = \$365;$$

$$r = \frac{4}{100} = 0.04;$$

$$t = 3.5;$$

$$i = ptr = 365 \times 3.5 \times 0.04 = \$51,10:$$

hence, $$S = 365 + 51,10 = \$416,10.$$

Present Value and Discount at Simple Interest.

The *present value* of any sum S, due t years hence, is the principal p, which put at interest for the time t, will produce the amount S.

The *discount* on any sum due t years hence, is the difference between that sum and the present value.

To find the present value of a sum of dollars denoted by S, *due* t *years hence, at simple interest, at the rate* r; also, *the discount.*

We have, from formula (2),

$$S = p + ptr;$$

and since p is the principal which in t years will produce the sum S, we have

$$p = \frac{S}{1+tr} \quad . \ . \ . \ (3);$$

and for the discount, which we will denote by D, we have

$$D = S - \frac{S}{1+tr} = \frac{Str}{1+tr} \quad . \ . \ . \ (4).$$

1. Required the discount on \$100, due 3 months hence, at the rate of $5\frac{1}{2}$ per cent. per annum.

$$S = \$100 \qquad = \$100$$
$$t = 3 \text{ months} = 0.25$$
$$r = \frac{5,5}{100} \qquad = 0.55.$$

Hence, the present value p is

$$p = \frac{S}{1+tr} = \frac{100}{1+.01375} = \$98,643:$$

hence, $\quad D = S - p = 100 - 98,643 = \$1,35,7.$

Compound Interest.

Compound interest is when the interest on a sum of money becoming due, and not paid, is added to the principal, and the interest then calculated on this amount as on a new principal.

To find the amount of a sum p *placed at interest for* t *years, compound interest being allowed annually at the rate* r.

At the end of one year the amount will be

$$S_{,} = p + pr = p(1+r).$$

Since compound interest is allowed, this sum now becomes the principal, and hence at the end of the second year the amount will be

$$S' = p(1+r) + pr(1+r) = p(1+r)^2.$$

Regard $p(1+r)^2$ as a new principal; we have, at the end of the third year,

$$S'' = p(1+r)^2 + pr(1+r)^2 = p(1+r)^3;$$

and at the end of t years,

$$S = p(1+r)^t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

And from Article 260 we have

$$\log. S = \log. p + t \log. (1+r);$$

and if any three of the four quantities S, p, t, and r, are given, the remaining one can be determined.

Let it be required to find the time in which a sum p will double itself at compound interest, the rate being 4 per cent. per annum.

We have from equation (5),

$$S = p(1+r)^t.$$

But by the conditions of the question,

$$S = 2p = p(1+r)^t:$$

hence, $\qquad 2 = (1+r)^t,$

and $\qquad t = \dfrac{\log. 2}{\log. (1+r)} = \dfrac{0.301030}{0.017033}$

$$= 17.673 \text{ years}$$

$$= 17 \text{ years, } 8 \text{ months, } 2 \text{ days.}$$

To find the Discount.

The discount being the difference between the sum S and p, we have

$$D = S - \frac{S}{(1+r)^t} = S\left(1 - \frac{1}{(1+r)^t}\right).$$

CHAPTER X.

GENERAL THEORY OF EQUATIONS.

274. THE most celebrated analysts have tried to resolve equations of any degree whatever, but hitherto their efforts have been unsuccessful with respect to equations of a higher degree than the fourth. However, their investigations have conducted them to some properties common to equations of every degree, which they have since used, either to resolve certain classes of equations, or to reduce the resolution of a given equation to that of one more simple. In this chapter it is proposed to make known these properties, and their use in facilitating the resolution of equations.

The development of the properties of equations of any degree, leads to the consideration of polynomials of a particular nature, and entirely different from those considered in the first chapter. These are expressions of the form

$$Ax^m + Bx^{m-1} + Cx^{m-2} + \ldots + Tx + U,$$

in which m is a positive whole number; but the co-efficients $A, B, C, \ldots T, U$, any quantities whatever, that is, entire or fractional, commensurable or incommensurable. Now, in algebraic division, as explained in Chapter II., the object was this, viz.: *having given two polynomials*, entire with reference to all the letters and particular numbers involved in them, *to find a third polynomial of the same kind, which multiplied by the second shall produce the first.*

But when we have two polynomials,

$$Ax^m + Bx^{m-1} + Cx^{m-2} + \ldots + Tx + U,$$
$$A'x^m + B'x^{m-1} + C'x^{m-2} + \ldots + T'x + U,$$

which are *necessarily* entire only with respect to x, and in which the co-efficients $A, B, C \ldots, A', B', C' \ldots$, are any quantities whatever, it may be proposed *to find a third polynomial*, of

the same form and nature as those that are given, *which multiplied by the second will re-produce the first.*

275. Ordinary polynomials, that is, polynomials which are entire with reference to all the exponents and co-efficients, are called *rational and entire polynomials.* Polynomials which are only entire with reference to the letter x, and whose co-efficients are any quantities whatever, are called *entire functions of* x.

276. Every complete equation of the m^{th} degree, m being a positive whole number, may, by the transposition of terms, and by the division of both members by the co-efficient of x^m, be put under the form

$$x^m + Px^{m-1} + Qx^{m-2} + \ . \ . \ . \ + Tx + U = 0,$$

$P, Q, R \ . \ . \ ., T, U,$ being co-efficients taken in the most general algebraic sense.

Any expression, which substituted in place of x *satisfies the equation, that is, renders its first member equal to* 0, *is called a root of the equation.*

277. As every equation may be considered as the translation into algebraic language of the relations which exist between the given and unknown quantities of a problem, we are naturally led to suppose that, EVERY EQUATION *has at least one root.* We will admit this principle, which we shall have occasion to verify hereafter for most equations.

We will now demonstrate some of the principal properties of a general equation.

First Property.

278. In every general equation of the form

$$x^m + Px^{m-1} + Qx^{m-2} + \ . \ . \ . \ + Tx + U = 0,$$

the first member is divisible by the difference between the unknown quantity x and a root of the equation; that is,

If a *is a root of the equation, the first member will be exactly divisible by* x — a; and reciprocally, *if a divisor of the form* x — a *will exactly divide the first member,* a *will be a root of the equation.*

Let us suppose the first member of the proposed equation to be divided by $x - a$, and the operation continued until all the terms

involving x are exhausted: the remainder, if there be any, will then be independent of x.

If we represent the remainder by R, and the quotient obtained by Q', we may write

$$x^m + Px^{m-1} \dots + Tx + U = Q'(x-a) + R.$$

Now, since by hypothesis, a is a root of the equation, if we substitute a for x, the first member of the equation will reduce to zero; the term $Q'(x-a)$ will also reduce to 0, and consequently, we shall have

$$R = 0.$$

But since R does not contain x, its value will not be affected by attributing to x the particular value a: hence, the remainder R was originally equal to zero, and consequently, the first member of the equation

$$x^m + Px^{m-1} + Qx^{m-2} \dots + Tx + U = 0,$$

is exactly divisible by $x - a$.

Reciprocally, if $x - a$ is an exact divisor of the first member of the equation, the quotient Q' will be exact, and we shall have $R = 0$: hence,

$$x^m + Px^{m-1} \dots + Tx + U = Q'(x-a).$$

If now, we suppose $x = a$, the second member will reduce to zero, consequently, the first will reduce to zero, and hence a will be a root of the equation (Art. 276). It is evident, from the nature of division, that the quotient Q' will be of the form

$$x^{m-1} + P'x^{m-2} \dots\dots\dots\dots + R' + U' = 0.$$

279. It follows from what has preceded, that in order to discover whether any polynomial is exactly divisible by the binomial $x - a$, it is sufficient to see if the substitution of a for x will reduce the polynomial to zero.

Reciprocally, if any polynomial is exactly divisible by $x - a$, then we know, that if the polynomial be placed equal to zero, x will be a root of the equation.

The property which we have demonstrated above, enables us to diminish the degree of an equation by unity when we know one of its roots, by a simple division; and if two or more of the roots are known, the degree of the equation may be still further diminished by continuing the division.

EXAMPLES.

1. One of the roots of the equation

$$x^4 - 25x^2 + 60x - 36 = 0$$

is 3: what is the equation containing the other roots?

$$\begin{array}{l|l} x^4 - 25x^2 + 60x - 36 & x - 3 \\ \cline{2-2} x^4 - 3x^3 & x^3 + 3x^2 - 16x + 12 \\ \hline + 3x^3 - 25x^2 & \\ 3x^3 - 9x^2 & \\ \hline + 16x^2 + 60x & \\ - 16x^2 - 48x & \\ \hline 12x - 36 & \\ 12x - 36 & \\ \hline \end{array}$$

Ans. $x^3 + 3x^2 - 16x + 12 = 0$.

2. Two roots of the equation

$$x^4 - 12x^3 + 48x^2 - 68x + 15 = 0$$

are 3 and 5: what is the equation containing the other two?

Ans. $x^2 - 4x + 1 = 0$.

3. One of the roots of the equation

$$x^3 - 6x^2 + 11x - 6 = 0$$

is 1: what is the equation containing the other roots?

Ans. $x^2 - 5x + 6 = 0$.

4. Two of the roots of the equation

$$4x^4 - 14x^3 - 5x^2 + 31x + 6 = 0$$

are 2 and 3: find the equation containing the other roots.

Ans. $4x^2 + 6x + 1 = 0$.

Second Property.

280. *Every equation involving but one unknown quantity, has as many roots as there are units in the exponent which denotes its degree, and no more.*

Let the proposed equation be

$$x^m + Px^{m-1} + Qx^{m-2} + \ldots + Tx + U = 0.$$

Since every equation is supposed to have at least one root (Art. 277), if we denote that root by a, the first member will be divisible by $x - a$, and we shall have the equation

$$x^m + Px^{m-1} + \ldots = (x - a)(x^{m-1} + P'x^{m-2} + \ldots) \ldots (1).$$

But if we place

$$x^{m-1} + P'x^{m-2} + \ldots = 0,$$

we obtain an equation which has at least one root.

Denote this root by b, we have (Art. 278),

$$x^{m-1} + P'x^{m-2} + \ldots = (x - b)(x^{m-2} + P''x^{m-3} + \ldots).$$

Substituting the second member, for its value in equation (1), and we have,

$$x^m + Px^{m-1} + \ldots = (x - a)(x - b)(x^{m-2} + P''x^{m-3} + \ldots) \ldots (2).$$

Reasoning upon the polynomial

$$x^{m-2} + P''x^{m-3} + \ldots$$

as upon the preceding polynomial, we have

$$x^{m-2} + P''x^{m-3} + \ldots = (x - c)(x^{m-3} + P'''x^{m-4} + \ldots),$$

and by substitution,

$$x^m + Px^{m-1} + \ldots = (x - a)(x - b)(x - c)(x^{m-3} + P'''x^{m-4}) \quad (3)$$

281. Observe, that for each binomial factor of the first degree with reference to x, the degree of x in the polynomial is diminished by unity; therefore, after $m - 2$ factors of the first degree have been divided out, the exponent of x will be reduced to $m - (m - 2) = 2$; that is, we shall obtain a polynomial of the second degree with reference to x, which can be decomposed into two factors of the first degree (Art. 142), of the form $x - k$, $x - l$. Now, supposing the $m - 2$ factors of the first degree to have already been indicated, we shall have the identical equation,

$$x^m + Px^{m-1} + \ldots = (x - a)(x - b)(x - c)\ldots(x - k)(x - l) = 0;$$

from which we see, that the *first member of the proposed equation may be decomposed into* m *binomial factors of the first degree.*

As there is *a root* corresponding to each binomial divisor of the first degree (Art. 278), it follows that the m binomial factors of the first degree, $x - a$, $x - b$, $x - c \ldots$, give the m roots, a, b, $c \ldots$, of the proposed equation.

But the equation can have no other roots than $a, b, c \ldots k, l$. For, if it had a root a', different from $a, b, c \ldots l$, it would have a divisor $x - a'$, different from $x - a$, $x - b$, $x - c \ldots x - l$, which is impossible. Therefore, finally,

Every equation of the mth *degree has* m *roots, and can have no more.*

282. In equations which arise from the multiplication of equal factors, such as

$$(x-a)^4(x-b)^3(x-c)^2(x-d)=0,$$

the number of roots is *apparently* less than the number of units in the exponent which denotes the degree of the equation. But this is not really so; for, the above equation actually has ten roots, four of which are equal to a, three to b, two to c, and one to d.

It is evident that no quantity a', different from a, b, c, d, can verify the equation; for, if it had a root a', the first member would be divisible by $x-a'$, which is impossible.

Consequence of the second Property.

283. It has been shown that the first member of every equation of the m^{th} degree, has m binomial divisors of the first degree, of the form

$$x-a, \quad x-b, \quad x-c, \ldots x-k, \quad x-l.$$

If we multiply these divisors together, *two and two, three and three*, &c., we shall obtain as many divisors of the second, third, &c. degree, with reference to x, as we can form different combinations of m quantities, taken two and two, three and three, &c. Now the number of these combinations is expressed by

$$m\cdot\frac{m-1}{2},\quad m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\ldots \text{(Art. 201).}$$

Hence, the proposed equation has

$$m\cdot\frac{m-1}{2}$$

divisors of the second degree;

$$m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}$$

divisors of the third degree;

$$m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot\frac{m-3}{4}$$

divisors of the fourth degree; and so on.

Composition of Equations.

284. If in the identical equation

$$x^m + Px^{m-1} + \ldots = (x-a)(x-b)(x-c)\ldots(x-l),$$

we perform the multiplication of four factors in the second member, we have,

$$\left.\begin{array}{l|l|l|l} x^4 - a & x^3 + ab & x^2 - abc & x + abcd \\ \quad\;\; - b & \quad\;\; + ac & \quad\;\; - abd & \\ \quad\;\; - c & \quad\;\; + ad & \quad\;\; - acd & \\ \quad\;\; - d & \quad\;\; + bc & \quad\;\; - bcd & \\ & \quad\;\; + bd & & \\ & \quad\;\; + cd & & \end{array}\right\} = 0.$$

If we perform the multiplication of the m factors of the second member, and compare the terms of the two members, we shall find the following relations between the co-efficients $P, Q, R, \ldots T, U$, and the roots $a, b, c, \ldots k, l$, of the proposed equation, viz.,

$$-a-b-c\ldots-k-l=P, \text{ or } a+b+c+\ldots+k+l=-P;$$

$$ab+ac+\ldots+kl=Q, \quad \ldots\ldots\ldots\ldots\ldots$$

$$-abc-abd\ldots-ikl=R, \text{ or } abc+abd\ldots\ldots+ikl=-R;$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\pm abcd\ldots kl=U, \text{ or } abcd\ldots kl=\pm U.$$

The double sign has been placed in the last relation, because the product $-a \times -b \times -c \ldots \times -l$ will be *plus* or *minus* according as the degree of the equation is *even* or *odd*. Hence,

1st. The algebraic sum of the roots, taken with contrary signs, is equal to the co-efficient of the second term; or, the algebraic sum of the roots themselves, is equal to the co-efficient of the second term taken with a contrary sign.

2d. The sum of the products of the roots taken two and two, with their respective signs, is equal to the co-efficient of the third term.

3d. The sum of the products of the roots taken three and three, with their signs changed, is equal to the co-efficient of the fourth term; or the co-efficient of the fourth term, taken with a contrary sign, is equal to the sum of the products of the roots taken three and three; and so on.

4th. The product of all the roots, is equal to the last term; that is, the product of all the roots, taken with their respective signs, is equal to the last term of the equation, taken with its sign, *when the equation is of an even degree,* and with a contrary sign, *when the equation is of an odd degree. If one of the roots is equal to* 0, *the absolute term will be* 0.

The properties demonstrated (Art. 143), with respect to equations of the second degree, are only particular cases of the above.

Consequences.

1. If the co-efficient of the second term of an equation is equal to zero, the term will not appear in the equation; and the sum of the positive roots is equal to the sum of the negative roots.

2. Every commensurable root of an equation is a divisor of the last or absolute term.

EXAMPLES IN THE FORMATION OF EQUATIONS.

1. Form the equation whose roots are 2, 3, 5, and -6.

We have, by simply indicating the multiplication of the factors,

$$(x-2)(x-3)(x-5)(x+6)=0.$$

But the process may be shortened by detaching the co-efficients thus:

$$\begin{array}{rrrrrl}
1 & -2 & & & & \underline{|-3} \\
 & -3 & +6 & & & \\
\hline
1 & -5 & +6 & & & \underline{|-5} \\
 & -5 & +25 & -30 & & \\
\hline
1 & -10 & +31 & -30 & & |+6 \\
 & 6 & -60 & +186 & -180 & \\
\hline
1 & -4 & -29 & +156 & -180. &
\end{array}$$

Hence, the required equation is

$$x^4-4x^3-29x^2+156x-180=0.$$

2. What is the equation whose roots are 1, 2, and -3?

Ans. $x^3-7x+6=0$.

3. What is the equation whose roots are 3, -4, $2+\sqrt{3}$, and $2-\sqrt{3}$? *Ans.* $x^4-3x^3-15x^2+49x-12=0$.

4. What is the equation whose roots are $3+\sqrt{5}$, $3-\sqrt{5}$, and -6? *Ans.* $x^3-32x+24=0$.

5. What is the equation whose roots are 1, -2, 3, -4, 5, and -6?

Ans. $x^6 + 3x^5 - 41x^4 - 87x^3 + 400x^2 + 444x - 720 = 0.$

6. What is the equation whose roots are . . . . $2 + \sqrt{-1}$, $2 - \sqrt{-1}$, and -3? *Ans.* $x^3 - x^2 + 7x + 15 = 0.$

Of the greatest Common Divisor.

285. The greatest common divisor of two polynomials is the greatest polynomial, with reference to its exponents and co-efficients, that will exactly divide the proposed polynomials.

If two polynomials be divided by their greatest common divisor, the quotients will be *prime with respect to each other*; that is, they will no longer contain a common factor. Hence,

Two polynomials are prime with respect to each other when they have not a common factor.

Let A and B be two polynomials, D their greatest common divisor, and A', B', the quotients after division. Then

$$\frac{A}{D} = A', \text{ and } \frac{B}{D} = B';$$

and consequently,

$$A = A' \times D, \text{ and } B = B' \times D.$$

Now, if A' and B' have a common factor d, then $d \times D$ would be a common divisor of the two polynomials and greater than D, either with respect to the exponents or the co-efficients, which would be contrary to the supposition.

Again, since D exactly divides A and B, every factor of D will have a corresponding factor in both A and B. Hence,

1st. *The greatest common divisor of two polynomials contains as factors, all the prime factors common to the two polynomials, and does not contain any others.*

286. We will now show that the greatest common divisor of two polynomials will divide their remainder after they have been divided by each other.

Let A and B be two polynomials, D their greatest common divisor, and suppose A to contain the highest exponent of the letter with reference to which the polynomials A and B are arranged.

Having divided A by B, suppose we have a quotient Q and a remainder R. We may then write

$$A = B \times Q + R.$$

If now, we divide both members of the equation by D, we have

$$\frac{A}{D} = \frac{B}{D} \times Q + \frac{R}{D};$$

and since we suppose A to be divisible by D, the first member of the equation will be entire, and consequently, the second member must also be entire, since an entire quantity cannot be equal to a fraction. But since D also divides B, the first term of the second member is entire, and consequently, the second term is also entire, and therefore, R is exactly divisible by D.

We will now show that if D will exactly divide B and R, that it will also divide A. For, having divided A by B, as before, we have

$A = B \times Q + R$, and by dividing by D, we obtain

$$\frac{A}{D} = \frac{B}{D} \times Q + \frac{R}{D}.$$

But since we suppose B and R to be divisible by D, and know Q to be an entire quantity, the second member of the equation is entire: hence, the first member is also entire, that is, A is exactly divisible by D. Hence,

2dly. *The greatest common divisor of two polynomials, is the same as that which exists between the least polynomial and their remainder after division.*

REMARK.—If either of the polynomials A or B have a factor A' common to all its terms, but not common to the other polynomial, the common divisor will be found in that part of the polynomial which is multiplied by the factor A'.

287. From these principles, we have, for finding the greatest common divisor of two polynomials, the following

RULE.

I. *Take the first polynomial and suppress all the monomial factors common to each of its terms. Do the same with the second polynomial, and if the factors so suppressed have a common divisor, set it aside as forming a part of the common divisor sought.*

II. *Having done this, prepare the dividend in such a manner that its first term shall be divisible by the first term of the divisor; then perform the division, and suppress in the remainder all the factors that are common to the co-efficients of the principal letter. Then take this remainder as a divisor, and the second polynomial as a dividend, and continue the operation with these polynomials, in the same manner as with the preceding.*

III. *Continue this series of operations until a remainder is obtained which will exactly divide the preceding divisor: this last divisor will be the greatest common divisor; but if a remainder is obtained which is independent of the principal letter, and which will not divide the co-efficients of each of the proposed polynomials, it shows that the proposed polynomials are prime with respect to each other, or that they have not a common factor.*

EXAMPLES.

1. Find the greatest common divisor of the polynomials

$$a^3 - a^2b + 3ab^2 - 3b^3, \quad \text{and} \quad a^2 - 5ab + 4b^2.$$

First Operation. *Second Operation.*

$$\begin{array}{ll|l} & a^3 - a^2b + 3ab^2 - 3b^3 & a^2 - 5ab + 4b^2 \\ \hline & \quad 4a^2b - \ ab^2 - 3b^3 & a + 4b \\ \hline \text{1st rem.} & 19ab^2 - 19b^3 & \\ \text{or,} & 19b^2(a - b). & \end{array}
\qquad
\begin{array}{c|l} a^2 - 5ab + 4b^2 & a - b \\ \hline -4ab + 4b^2 & a - 4b \\ \hline 0. & \end{array}$$

Hence, $a - b$ is the greatest common divisor.

We begin by dividing the polynomial of the highest degree by that of the lowest; the quotient is, as we see in the above table, $a + 4b$, and the remainder $19ab^2 - 19b^3$.

But, $$19ab^2 - 19b^3 = 19b^2(a - b).$$

Now, the factor $19b^2$, will divide this remainder without dividing

$$a^2 - 5ab + 4b^2:$$

hence, the factor must be suppressed, and the question is reduced to finding the greatest common divisor between

$$a^2 - 5ab + 4b^2 \quad \text{and} \quad a - b.$$

Dividing the first of these two polynomials by the second, there is an exact quotient, $a - 4b$; hence, $a - b$ is the greatest common divisor of the two given polynomials. To verify this, let each be divided by $a - b$.

2. Find the greatest common divisor of the polynomials

$$3a^5 - 5a^3b^2 + 2ab^4 \quad \text{and} \quad 2a^4 - 3a^2b^2 + b^4.$$

We first suppress a, which is a factor of each term of the first polynomial: we then have

$$3a^4 - 5a^2b^2 + 2b^4 \,\|\, 2a^4 - 3a^2b^2 + b^4.$$

We now find that the first term of the dividend will not contain the first term of the divisor. We therefore multiply the dividend by 2, which merely introduces into the dividend a factor not common to the divisor, and hence does not affect the common divisor sought. We then have

$$\begin{array}{r|l} 6a^4 - 10a^2b^2 + 4b^4 & 2a^4 - 3a^2b^2 + b^4 \\ \cline{2-2} 6a^4 - 9a^2b^2 + 3b^4 & 3 \\ \hline - a^2b^2 + b^4 & \\ - b^2(a^2 - b^2). & \end{array}$$

We find after division, the remainder $-a^2b^2 + b^4$, which we put under the form $-b^2(a^2 - b^2)$. We then suppress $-b^2$, and divide

$$\begin{array}{r|l} 2a^4 - 3a^2b^2 + b^4 & a^2 - b^2 \\ \cline{2-2} 2a^4 - 2a^2b^2 & 2a^2 - b^2 \\ \hline - a^2b^2 + b^4 & \\ - a^2b^2 + b^4. & \\ \hline \end{array}$$

Hence, $a^2 - b^2$ is the greatest common divisor.

3. Let it be required to find the greatest common divisor between the two polynomials

$$-3b^3 + 3ab^2 - a^2b + a^3, \quad \text{and} \quad 4b^2 - 5ab + a^2.$$

First Operation.

$$\begin{array}{lr|l} & -12b^3 + 12ab^2 - 4a^2b + 4a^3 & 4b^2 - 5ab + a^2 \\ \cline{3-3} \text{1st rem.} \;\cdot\;\cdot & -3ab^2 - a^2b + 4a^3 & -3b, \quad -3a \\ & -12ab^2 - 4a^2b + 16a^3 & \\ \cline{2-2} \text{2d rem.} \;\cdot\;\cdot & -19a^2b + 19a^3 & \\ \text{or,} & 19a^2(-b + a). & \end{array}$$

Second Operation.

$$\begin{array}{r|l} 4b^2 - 5ab + a^2 & -b + a \\ \cline{2-2} \hline - ab + a^2 & -4b + a \\ \hline 0. & \end{array}$$

Hence, $-b + a$, or $a - b$, is the greatest common divisor.

In the first operation we meet with a difficulty in dividing the two polynomials, because the first term of the dividend is not exactly divisible by the first term of the divisor. But if we observe that the co-efficient 4, is not a factor of all the terms of the polynomial

$$4b^2 - 5ab + a^2,$$

and therefore, by the first principle, that 4 cannot form a part of the greatest common divisor, we can, without affecting this common divisor, introduce this factor into the dividend. This gives

$$-12b^3 + 12ab^2 - 4a^2b + 4a^3,$$

and then the division of the terms is possible.

Effecting this division, the quotient is $-3b$, and the remainder is

$$-3ab^2 - a^2b + 4a^3.$$

As the exponent of b in this remainder is still equal to that of b in the divisor, the division may be continued, by multiplying this remainder by 4, in order to render the division of the first term possible. This done, the remainder becomes

$$-12ab^2 - 4a^2b + 16a^3,$$

which divided by $4b^2 - 5ab + a^2$, gives the quotient $-3a$, which should be separated from the first by a comma, having no connexion with it. The remainder after this division is

$$-19a^2b + 19a^3.$$

Placing this last remainder under the form $19a^2(-b + a)$, and suppressing the factor $19a^2$, as forming no part of the common divisor, the question is reduced to finding the greatest common divisor between

$$4b^2 - 5ab + a^2 \quad \text{and} \quad -b + a.$$

Dividing the first of these polynomials by the second, we obtain an exact quotient, $-4b + a$: hence, $-b + a$, or $a - b$, is the greatest common divisor sought.

288. In the above example, as in all those in which the exponent of the principal letter is greater by unity in the dividend than in the divisor, we can abridge the operation by first multiplying every term of the dividend by the square of the co-effi-

cient of the first term of the divisor. We can easily see that by this means, the first partial quotient obtained will contain the first power of this co-efficient. Multiplying the divisor by the quotient, and making the reductions with the dividend thus prepared, the result will still contain the co-efficient as a factor, and the division can be continued until a remainder is obtained of a lower degree than the divisor, with reference to the principal letter.

Take the same example as before, viz.,

$$-3b^3 + 3ab^2 - a^2b + a^3 \quad \text{and} \quad 4b^2 - 5ab + a^2,$$

and multiply the dividend by the square of $4 = 16$; and we have

First Operation.

$$\begin{array}{l|l} -48b^3 + 48ab^2 - 16a^2b + 16a^3 & 4b^2 - 5ab + a^2 \\ \hline \quad\quad\quad - 12ab^2 - \ 4a^2b + 16a^3 & -12b - 3a \\ \hline \end{array}$$

1st. remainder, $\quad -19a^2b + 19a^3$

or, $\quad 19a^2 \ (-b + a).$

Second Operation.

$$\begin{array}{l|l} 4b^2 - 5ab + a^2 & - \ b + a \\ \hline - \ ab + a^2 & -4b + a \\ \hline \end{array}$$

2d remainder, $\quad -0.$

289. When the exponent of the principal letter in the dividend exceeds that of the same letter in the divisor by two, three, &c. units, multiply the dividend by the third, fourth, &c. power of the co-efficient of the first term of the divisor. It is easy to see the reason of this.

It may be asked if the suppression of the factors, common to all the terms of one of the remainders, is *absolutely necessary*, or whether the object is merely to render the operations more simple. It will easily be perceived that the suppression of these factors *is necessary;* for, if the factor $19a^2$ was not suppressed in the preceding example, it would be necessary to multiply the whole dividend by this factor, in order to render its first term divisible by the first term of the divisor; but then, a factor would be introduced into tne dividend which is also contained in the divisor; and consequently, the required greatest common divisor would be combined with the factor $19a^2$, which forms no part of it.

290. For another example, let it be required to find the greatest common divisor between the two polynomials,

$$a^4 + 3a^3b + 4a^2b^2 - 6ab^3 + 2b^4 \quad \text{and} \quad 4a^2b + 2ab^2 - 2b^3,$$

or simply,

$$2a^2 + ab - b^2,$$

since the factor $2b$ can be suppressed, being a factor of the second polynomial and not of the first.

First Operation.

$$\begin{array}{lr|l}
 & 8a^4 + 24a^3b + 32a^2b^2 - 48ab^3 + 16b^4 & 2a^2 + ab - b^2 \\
 & + 20a^3b + 36a^2b^2 - 48ab^3 + 16b^4 & 4a^2 + 10ab + 13b^2 \\
 & + 26a^2b^2 - 38ab^3 + 16b^4 & \\
\text{1st remainder,} & - 51ab^3 + 29b^4 & \\
\text{or,} & - b^3(51a - 29b). &
\end{array}$$

Second Operation.

Multiply by 2601, the square of 51.

$$\begin{array}{lr|l}
 & 5202a^2 + 2601ab - 2601b^2 & 51a - 29b \\
 & 5202a^2 - 2958ab & 102a + 109b \\
\text{1st remainder,} & + 5559ab - 2601b^2 & \\
 & 5559ab - 3161b^2 & \\
\text{2d remainder,} & + 560b^2. &
\end{array}$$

The exponent of the letter a in the dividend, exceeding that of the same letter in the divisor by *two* units, the whole dividend is multiplied by the cube of $2 = 8$. This done, we perform three consecutive divisions, and obtain for the first principal remainder,

$$-51ab^3 + 29b^4.$$

Suppressing b^3, the remainder becomes, $-51a + 29b$; and changing the signs, which is permitted, we have $51a - 29b$; and the new dividend is

$$2a^2 + ab - b^2.$$

Multiplying the dividend by the square of $51 = 2601$, then effecting the division, we obtain for the second principal remainder, $+560b^2$. Now, it results from the second principle (Art. 286), that the greatest common divisor must be a factor of the remainder after each division; therefore it should divide the remainder

$560b^2$. But this remainder is *independent* of the principal letter a: hence, if the two polynomials have a common divisor, it must be *independent* of a, and will consequently be found as a factor in the co-efficients of the different powers of this letter, in each of the proposed polynomials. But it is evident that the co-efficients of these polynomials have not a common factor. Hence, *the two given polynomials are prime with respect to each other.*

Remarks on the greatest common Divisor.

291. The rule for finding the greatest common divisor of two polynomials, may readily be extended to three or more polynomials. For, having the polynomials A, B, C, D, &c., if we find the greatest common divisor of A and B, and then the greatest common divisor of this result and C, the divisor so obtained will evidently be the greatest common divisor of A, B, and C; and the same process may be applied to the remaining polynomials.

292. Let A be a rational and entire polynomial, supposed to be arranged with reference to one of the letters involved in it, a, for example.

If this polynomial is not *absolutely prime*, that is, if it can be decomposed into rational and entire factors, it may be regarded as the product of three principal factors, viz.,

1st. Of a monomial factor A', common to all the terms of A. This factor is composed of the greatest common divisor of all the numerical co-efficients, multiplied by the product of the literal factors which are common to all the terms of the polynomial.

2d. Of a polynomial factor A'', independent of a, which is common to all the co-efficients of the different powers of a, in the arranged polynomial.

3d. Of a polynomial factor A''', depending upon a, and in which the co-efficients of the different powers of a are prime with each other, so that we shall have

$$A = A' \times A'' \times A'''.$$

Sometimes, one or both of the factors A', A'', reduce to unity; but the above is the general form of *rational and entire* polynomials. Hence, their greatest common divisor may assume the form

$$D = D' \times D'' \times D''';$$

D' denoting the greatest monomial common factor, D'' the greatest polynomial factor independent of a, and D''' the greatest polynomial factor depending upon this letter.

In order to obtain D', *find the monomial factor A' common to all the terms of A.* This factor is in general composed of literal factors, which are found by inspecting the terms, and of a numerical co-efficient, obtained by finding the greatest common divisor of the numerical co-efficients in A.

In the same way, find the monomial B' common to all the terms of B; then *determine the greatest factor D', common to A' and B'.*

This factor D' is set aside, as forming the first part of the required common divisor. *The factors A' and B' are also suppressed in the proposed polynomials,* and the question is reduced to finding the greatest common divisor of two new polynomials which do not contain a common monomial factor.

EXAMPLES.

1. It is required to find the greatest common divisor of the two polynomials

$$a^2d^2 - c^2d^2 - a^2c^2 + c^4, \quad \text{and} \quad 4a^2d - 2ac^2 + 2c^3 - 4acd.$$

The second contains a monomial factor 2. Suppressing it, and arranging the polynomials with reference to d, we have

$$(a^2 - c^2)\,d^2 - a^2c^2 + c^4, \quad \text{and} \quad (2a^2 - 2ac)\,d - ac^2 + c^3.$$

It is first necssary to ascertain whether there is a common divisor independent of d.

By considering the co-efficients $a^2 - c^2$ and $-a^2c^2 + c^4$, of the first polynomial, it will be seen that $-a^2c^2 + c^4$ can be put under the form $-c^2(a^2 - c^2)$: hence $a^2 - c^2$ is a common factor of the co-efficients of the first polynomial. In like manner, the co-efficients of the second, $2a^2 - 2ac$ and $-ac^2 + c^3$, can be reduced to $2a(a - c)$ and $-c^2(a - c)$; therefore, $a - c$ is a common factor of these co-efficients.

Comparing the two factors $a^2 - c^2$ and $a - c$, we see that the last will divide the first; hence it follows that $a - c$ is a common factor of the proposed polynomials, and it is that part of their greatest common divisor *which is independent of* d.

Suppressing $a^2 - c^2$ in the first polynomial, and $a - c$ in the second, we obtain the two polynomials $d^2 - c^2$ and $2ad - c^2$, to which the ordinary process must be applied

$$\begin{array}{r|l}
d^2 - c^2 & \underline{2ad - c^2} \\
4a^2d^2 - 4a^2c^2 & 2ad + c^2 \\
\hline
+ 2ac^2d - 4a^2c^2 & \\
\hline
- 4a^2c^2 + c^4. &
\end{array}$$

After having multiplied the dividend by $4a^2$, and performed two consecutive divisions, we obtain a remainder $-4a^2c^2 + c^4$, independent of the letter d: hence the two polynomials $d^2 - c^2$ and $2ad - c^2$, are prime with each other. Therefore, the greatest common divisor of the proposed polynomials is $a - c$.

Again, taking the same example, and arranging with reference to a, it becomes, after suppressing the factor 2 in the second polynomial,

$$(d^2 - c^2)\,a^2 - c^2d^2 + c^4, \quad \text{and} \quad 2da^2 - (2cd + c^2)\,a + c^3.$$

It is easily perceived, that the co-efficient of the different powers of a in the second polynomial, are prime with each other. In the first polynomial, the co-efficient $-c^2d^2 + c^4$, of the second term, or of a^0, becomes $-c^2(d^2 - c^2)$; whence, $d^2 - c^2$ is a common factor of the two co-efficients, and since it is not a factor of the second polynomial, it may be suppressed in the first, as not forming a part of the common divisor.

By suppressing this factor, and taking the second polynomial for a dividend and the first for a divisor (in order to avoid preparation), we have

$$\begin{array}{lr|l|l}
\text{1st.} & 2da^2 - 2cd & a + c^3 & \underline{a^2 - c^2} \\
 & - c^2 & & 2d \\
\hline
\text{Rem.} & - 2cd & a + 2dc^2 & \\
 & - c^2 & + \quad c^3 & \\
\text{or,} & & a - c, &
\end{array}$$

by suppressing the common factor $(-2cd - c^2)$.

$$\begin{array}{lr|l}
\text{2d.} & a^2 - c^2 & \underline{a - c} \\
 & + ac - c^2 & a + c \\
\hline
 & 0 &
\end{array}$$

After having performed the first division, a remainder is obtained which contains $-2cd - c^2$, as a factor of its two co-

efficients; for $2dc^2 + c^3 = -c(-2cd - c^2)$. This factor being suppressed, the remainder is reduced to $a - c$, which will exactly divide $a^2 - c^2$.

Hence, $a - c$ is the required greatest common divisor.

293. There is a remarkable case, in which the greatest common divisor may be obtained more easily than by the general method; it is, when *one of the two polynomials contains a letter which is not contained in the other.*

In this case, it is evident, that the greatest common divisor is independent of this letter. Hence, by arranging the polynomial which contains it, with reference to this letter, *the required common divisor will be the same as that which exists between the co-efficients of the different powers of the principal letter and the second polynomial.*

By this method we are led, it is true, to determine the greatest common divisor between three or more polynomials. But they will be more simple than the proposed polynomials. It often happens, that some of the co-efficients of the arranged polynomial are monomials, or, that we can discover by simple inspection that they are prime with each other; and, in this case, we are certain that the proposed polynomials are prime with each other.

Thus, in the example 1, treated by the first method, after having suppressed the common factor $a - c$, which gives the results,

$$d^2 - c^2 \quad \text{and} \quad 2ad - c^2,$$

we know immediately that these two polynomials are prime with each other; for, since the letter a is contained in the second and not in the first, it follows from what has just been said, that the common divisor must divide the co-efficients $2d$ and $-c^2$, which is evidently impossible; hence, they are prime with respect to each other.

2. Let it be required to find the greatest common divisor of the two polynomials,

$$3bcq + 30mp + 18bc + 5mpq$$

and

$$4adq - 42fg + 24ad - 7fgq,$$

by the last principle.

We observe, in the first place, that the two polynomials do not contain any common monomial factor.

Since q is common to the two polynomials, we can arrange them with reference to this letter, and follow the ordinary rule. But as b is found in the first polynomial and not in the second. If then, we arrange the first with reference to b, which gives

$$(3cq + 18c)\,b + 30mp + 5mpq,$$

the required greatest common divisor will be the same as that which exists between the second polynomial and the two co-efficients

$$3cq + 18c \quad \text{and} \quad 30mp + 5mpq.$$

Now, the first of these co-efficients can be put under the form $3c\,(q + 6)$, and the other becomes $5mp\,(q + 6)$; hence $q + 6$ is a common factor of these co-efficients. It will therefore be sufficient to ascertain whether $q + 6$, which is a *prime* divisor, is a factor of the second polynomial.

Arranging this polynomial with reference to q, it becomes

$$(4ad - 7fg)\,q - 42fg + 24ad;$$

as the second part, $24ad - 42fg = 6\,(4ad - 7fg)$, it follows that this polynomial is divisible by $q + 6$, and gives the quotient $4ad - 7fg$. Therefore, $q + 6$ is the greatest common divisor of the proposed polynomials.

REMARK.—It may be ascertained that $q + 6$ is an exact divisor of the polynomial

$$(4ad - 7fg)\,q + 24ad - 42fg,$$

by a method derived from the property proved in Art. 278.

Make $q + 6 = 0$, or $q = -6$, in this polynomial; it becomes

$$(4ad - 7fg) \times -6 + 24ad - 42fg = 0;$$

that is, -6 substituted for q reduces the polynomial to 0; hence $q + 6$ is a divisor of this polynomial.

This method may be advantageously employed in nearly all the applications of the process. It consists in this, viz., after obtaining a remainder of the first degree with reference to a, when a is the principal letter, *make this remainder equal to* 0, *and deduce the value of* a *from this equation.*

If this value, substituted in the remainder of the 2d degree, *destroys it*, then the remainder of the 1st degree, simplified Art. 292, is a common divisor. If the remainder of the 2d degree

does not reduce to 0 by this substitution, we may conclude that there is no common divisor depending upon the principal letter.

Farther, having obtained a remainder of the 2d degree, with reference to a, it is not necessary to continue the operation any farther. For,

Decompose this polynomial into two factors of the 1*st degree*, which is done by placing it equal to 0, and resolving the resulting equation of the 2d degree.

When each of the values of a thus obtained, substituted in the remainder of the 3d degree, *destroys it*, it is a proof that the remainder of the 2d degree, *simplified*, is a common divisor; when only one of the values destroys the remainder of the 3d degree, the common divisor is the factor of the 1st degree with respect to a, which corresponds to this value.

Finally, when neither of these values destroys the remainder of the 3d degree, we may conclude that there is not a common divisor depending upon the letter a.

It is here supposed that the two factors of the 1st degree with reference to a, are rational; otherwise it would be more simple to perform the division of the remainder of the 3d degree by that of the second, and when this last division cannot be performed exactly, we may be certain that there is no rational common divisor, for if there was one, it could only be of the 1st degree with respect to a, and should be found in the remainder of the 2d degree, which is contrary to the hypothesis.

3. Find the greatest common divisor of the two polynomials

$$6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1$$

and $$4x^4 + 2x^3 - 18x^2 + 3x - 5.$$

Ans. $2x^3 - 4x^2 + x - 1.$

4. Find the greatest common divisor of the polynomials

$$20x^6 - 12x^5 + 16x^4 - 15x^3 + 14x^2 - 15x + 4.$$

and $$15x^4 - 9x^3 + 47x^2 - 21x + 28.$$

Ans. $5x^2 - 3x + 4.$

5. Find the greatest common divisor of the two polynomials

$$5a^4b^2 + 2a^3b^3 + ca^2 - 3a^2b^4 + bca$$

and $$a^5 + 5a^3d - a^3b^2 + 5a^2bd.$$

Ans. $a^2 + ab.$

Transformation of Equations.

The transformation of an equation consists in changing its form without affecting the equality of its members. The object of a transformation, is to change an equation from a given form, to another form that is more easily resolved.

First Transformation.

To make the Denominators disappear from an Equation.

294. If we have an equation of the form

$$x^m + Px^{m-1} + Qx^{m-2} + \ldots Tu + U = 0,$$

and make
$$x = \frac{y}{k};$$

we shall have, after substituting this value for x, and multiplying every term by k^m,

$$y^m + Pky^{m-1} + Qk^2y^{m-2} + Rk^3y^{m-3} + \ldots + Tk^{m-1}y + Uk^m = 0,$$

an equation of which the co-efficients are equal to those of the given equation, multiplied respectively by k^0, k^1, k^2, k^3, k^4, &c.

This transformation is principally used *to make the denominators disappear from an equation, when the co-efficient of the first term is unity.*

As an example, take the equation of the 4th degree,

$$x^4 + \frac{a}{b}x^3 + \frac{c}{d}x^2 + \frac{e}{f}x + \frac{g}{h} = 0.$$

If we make
$$x = \frac{y}{k},$$

y being a new unknown quantity and k an indeterminate quantity, we have

$$y^4 + \frac{ak}{b}y^3 + \frac{ck^2}{d}y^2 + \frac{ek^3}{f}y + \frac{gk^4}{h} = 0.$$

Now, there may be two cases—

1st. Where the denominators b, d, f, h, are prime with each other. In this hypothesis, as k is altogether arbitrary, take $k = bdfh$, *the product of the denominators*, the equation will then become

$$y^4 + adfh \,.\, y^3 + cb^2df^2h^2 \,.\, y^2 + eb^3d^3f^2h^3 \,.\, y + gb^4d^4f^4h^3 = 0,$$

in which the co-efficients are entire, and that of its first term unity.

We can determine the values of x corresponding to those of y from the equation,

$$x = \frac{y}{bdfh}.$$

2d. When the denominators contain common factors, we shall evidently render the co-efficients entire, by making k equal to the smallest multiple of all the denominators. But we can simplify still more, by giving to k such a value that k^1, k^2, k^3, . . . shall contain the prime factors which compose b, d, f, h, raised to powers at least equal to those which are found in the denominators.

Thus, the equation

$$x^4 - \frac{5}{6}x^3 + \frac{5}{12}x^2 - \frac{7}{150}x - \frac{13}{9000} = 0,$$

becomes

$$y^4 - \frac{5k}{6}y^3 + \frac{5k^2}{12}y^2 - \frac{7k^3}{150}y - \frac{13k^4}{9000} = 0,$$

after making $x = \frac{y}{k}$, and reducing to entire terms.

First, if we make $k = 9000$, which is a multiple of all the other denominators, it is clear, that the co-efficients become whole numbers.

But if we decompose 6, 12, 150, and 9000, into their factors, we find

$$6 = 2 \times 3, \quad 12 = 2^2 \times 3, \quad 150 = 2 \times 3 \times 5^2, \quad 9000 = 2^3 \times 3^2 \times 5^3;$$

and by simply making

$$k = 2 \times 3 \times 5,$$

the product of the different simple factors, we obtain

$$k^2 = 2^2 \times 3^2 \times 5^2, \quad k^3 = 2^3 \times 3^3 \times 5^3, \quad k^4 = 2^4 \times 3^4 \times 5^4;$$

whence we see that the values of k, k^2, k^3, k^4, contain the prime factors of 2, 3, 5, raised to powers at least equal to those which enter into 6, 12, 150, and 9000.

Hence, the hypothesis

$$k = 2 \times 3 \times 5,$$

is sufficient to make the denominators disappear. Substituting this value, the equation becomes

$$y^4 - \frac{5.2.3.5}{2.3}y^3 + \frac{5.2^2.3^2.5^2}{2^2.3}y^2 - \frac{7.2^3.3^3.5^3}{2.3.5^2}y - \frac{13.2^4.3^4.5^4}{2^3.3^2.5^3} = 0,$$

which reduces to

$$y^4 - 5.5y^3 + 5.3.5^2y^2 - 7.2^2.3^2.5y - 13.2.3^2.5 = 0;$$

or $$y^4 - 25y^3 + 375y^2 - 1260y - 1170 = 0.$$

Hence, we perceive the necessity of taking k as small a number as possible: otherwise, we should obtain a transformed equation, having its co-efficients very great, as may be seen by reducing the transformed equation resulting from the supposition $k = 9000$.

Hence we see, *that any equation may be transformed into another equation, of which the roots shall be a multiple or sub-multiple of those of the given equation.*

EXAMPLES.

1. $$x^3 - \frac{7}{3}x^2 + \frac{11}{36}x - \frac{25}{72} = 0.$$

Making $x = \frac{y}{6}$, and we have

$$y^3 - 14y^2 + 11y - 75 = 0.$$

2. $$x^5 - \frac{13}{12}x^4 + \frac{21}{40}x^3 - \frac{32}{225}x^2 - \frac{43}{600}x - \frac{1}{800} = 0.$$

Making $x = \frac{y}{2^2.3.5} = \frac{y}{60}$, and we have

$$y^5 - 65y^4 + 1890y^3 - 30720y^2 - 928800y + 972000 = 0.$$

Second Transformation.

To make the second Term disappear from an Equation.

295. The difficulty of resolving an equation generally diminishes with the number of terms involving the unknown quantity. Thus the equation

$$x^2 = q, \quad \text{gives immediately,} \quad x = \pm\sqrt{q}.$$

while the complete equation

$$x^2 + 2px + q = 0,$$

requires preparation before it can be resolved.

Now, any given equation can always be *transformed* into another equation, in which the second term shall be wanting.

For, let there be the general equation

$$x^m + Px^{m-1} + Qx^{m-2} + \ldots + Tx + U = 0.$$

Suppose $$x = u + x',$$

u being unknown, and x' an *indeterminate quantity*. By substituting $u + x'$ for x, we obtain

$$(u+x')^m + P(u+x')^{m-1} + Q(u+x')^{m-2} \ldots + T(u+x') + U = 0.$$

Developing by the binomial formula, and arranging according to the decreasing powers of u, we have

$$\left.\begin{array}{l|l|l} u^m + mx' & u^{m-1} + m\,.\dfrac{m-1}{2}x'^2 & u^{m-2} + \ldots + x'^m \\ \quad + P & \quad + (m-1)Px' & \qquad\qquad + Px'^{m-1} \\ & \quad + Q & \qquad\qquad + Qx'^{m-2} \\ & & \qquad\qquad + \ \ldots \\ & & \qquad\qquad \ldots\ldots \\ & & \qquad\qquad + Tx' \\ & & \qquad\qquad + U \end{array}\right\} = 0.$$

Since x' is entirely arbitrary, we may dispose of it in such a way that we shall have

$$mx' + P = 0; \quad \text{whence,} \quad x' = -\frac{P}{m}.$$

Substituting this value of x' in the last equation, we shall obtain an equation of the form,

$$u^m + Q'u^{m-2} + R'u^{m-3} + \ldots T'u + U' = 0,$$

in which the second term is wanting.

If this equation were resolved, we could obtain any value of x corresponding to that of u, from the equation

$$x = u + x', \quad \text{or} \quad x = u - \frac{P}{m}.$$

Whence, in order to make the second term of an equation disappear,

Substitute for the unknown quantity a new unknown quantity, united with the co-efficient of the second term, taken with a contrary sign, and divided by the exponent of the degree of the equation.

Let us apply the preceding rule to the equation

$$x^2 + 2px = q.$$

If we make $\qquad x = u - p,$

we have $\qquad (u-p)^2 + 2p(u-p) = q;$

and by performing the multiplications and reducing,

$$u^2 - p^2 = q,$$

which gives $$u = \pm\sqrt{q + p^2};$$

and consequently, $$x = -p \pm \sqrt{q + p^2}.$$

296. Instead of making the second term disappear, it may be required to find an equation which shall be deprived of its third, fourth, or any other term. This is done, by making the co-efficient of u corresponding to that term equal to 0. For example, to make the third term disappear, we make, in the above-transformed equation

$$m\frac{m-1}{2}x'^2 + (m-1)Px' + Q = 0,$$

from which we obtain two values for x', which substituted in the transformed equation reduce it to the form

$$u^m + P'u^{m-1} + R'u^{m-3} \ldots + T'u + U' = 0.$$

Beyond the third term it will be necessary to resolve an equation of a degree superior to the second, to obtain the value of x'; and to cause the last term to disappear, it will be necessary to resolve the equation

$$x'^m + Px'^{m-1} \ldots + Tx' + U = 0,$$

which is what the given equation becomes when x' is substituted for x.

It may happen that the value

$$x' = -\frac{P}{m},$$

which makes the second term disappear, causes also the disappearance of the third or some other term. For example, in order that the third term may disappear at the same time with the second, it is necessary that the value of x' which results from the equation

$$x' = -\frac{P}{m},$$

shall also satisfy the equation

$$m\frac{m-1}{2}x'^2 + (m-1)Px' + Q = 0.$$

Now, if in this last equation, we replace x' by $-\frac{P}{m}$, we have

$$m\frac{m-1}{2}\cdot\frac{P^2}{m^2} - (m-1)\frac{P^2}{m} + Q = 0, \quad \text{or} \quad (m-1)P^2 - 2mQ = 0;$$

and consequently, if

$$P^2 = \frac{2mQ}{m-1},$$

the disappearance of the second term will also involve that of the the third.

Formation of derived Polynomials.

297. The relation

$$x = u + x',$$

which has been used in the two preceding articles, indicates that the roots of the transformed equations are equal to those of the given equation, increased or diminished by a certain quantity. Sometimes this quantity is introduced into the calculus, as an indeterminate quantity, the value of which is afterward determined by requiring it to satisfy a given condition; sometimes it is a particular number, of a given value, which expresses a *constant difference* between the roots of a primitive equation and those of another equation which we wish to form.

In short, the transformation, which consists in substituting $u + x'$ for x, in a given equation, is of very frequent use in the theory of equations. There is a very simple method of obtaining, in practice, the transformation which results from this substitution.

To show this, let us substitute for x, $u + x'$ in the equation

$$x^m + Px^{m-1} + Qx^{m-2} + Rx^{m-3} + \ldots Tx + U = 0;$$

then, by developing, and arranging the terms according to the ascending powers of u, we have

$$\left.\begin{array}{ll|l|l}
x'^m & +mx'^{m-1} & u + m\dfrac{m-1}{1.2}x'^{m-2} & u^2 + \ldots u^m \\
+Px'^{m-1} & +(m-1)Px'^{m-2} & +(m-1)\dfrac{m-2}{1.2}Px'^{m-3} & \\
+Qx'^{m-2} & +(m-2)Qx'^{m-3} & +(m-2)\dfrac{m-3}{1.2}Qx'^{m-4} & \\
+\ldots\ldots & +\ldots & +\ldots & \\
+Tx' & +T & & \\
+U & & &
\end{array}\right\} = 0$$

If we observe how the co-efficients of the different powers of u are composed, we shall see that the co-efficient of u^0, is what the

first member of the given equation, becomes when x' is substituted in place of x; we shall denote this expression by X'.

The co-efficient of u^1 is formed from the preceding term X', by multiplying each term of X' by the exponent of x' in that term, and then diminishing this exponent by unity; we shall denote this co-efficient by Y'.

The co-efficient of u^2 is formed from Y', by multiplying each term of Y' by the exponent of x' in that term, dividing the product by 2, and then diminishing each exponent by unity. Representing this co-efficient by $\frac{Z'}{2}$, we see that *Z' is formed from Y', in the same manner that Y' is formed from X'.*

In general, the co-efficient of any power of u, in the above-transformed equation, may be found from the preceding co-efficient in the following manner: viz.,

By taking each term of that co-efficient in succession, multiplying it by the exponent of x′, dividing by the number which marks the place of the co-efficient, and diminishing the exponent of x′ by unity.

The law by which the co-efficient

$$X', \quad Y', \quad \frac{Z'}{1\,.\,2}, \quad \frac{V'}{1\,.\,2\,.\,3},$$

are derived from each other, is evidently the same as that which governs the formation of the terms of the binomial formula (Art. 203). The expressions,

$$Y', \quad Z', \quad V', \quad W' \; . \; . \; . \; .$$

are called *derived polynomials* of X', because each is derived from the one which precedes it, by the same law as that by which Y' is deduced from X'. Hence, generally,

A derived polynomial is one which is deduced from a given polynomial, according to a fixed and known law.

Recollect that X' is what the given polynomial becomes when x' is substituted for x.

Y' is called the *first-derived* polynomial;

Z' is called the *second-derived* polynomial;

V' is called the *third-derived* polynomial.

&c., &c.

We should also remember if we make $u = 0$, we shall have, $x' = x$, whence X' will become the given polynomial, from which the derived polynomials will then be obtained.

298. Let us now apply the above principles in the following

EXAMPLES.

1. Let it be required to find the derived polynomials from the equation

$$3x^4 + 6x^3 - 3x^2 + 2x + 1 = 0 = X.$$

Now, u being zero, and $x' = x$, we have from the law of forming the derived polynomials,

$$\begin{aligned} X = X' &= 3x^4 + 6x^3 - 3x^2 + 2x + 1; \\ Y' &= 12x^3 + 18x^2 - 6x + 2; \\ Z' &= 36x^2 + 36x - 6; \\ V' &= 72x + 36; \\ W' &= 72. \end{aligned}$$

It should be remarked that the exponent of x in the terms 1, 2 $-$ 6, 36, and 72, is equal to 0; hence, each of those terms disappears in the following derived polynomial.

2. Let it be required to cause the second term to disappear in the equation

$$x^4 - 12x^3 + 17x^2 - 9x + 7 = 0.$$

Make (Art. 295), $\quad x = u + \dfrac{12}{4} = u + 3;$

whence, $\quad x' = 3.$

The transformed equation will be of the form

$$X' + Y'u + \frac{Z'}{2}u^2 + \frac{V'}{2 \times 3}u^3 + u^4 = 0,$$

and the operation is reduced to finding the values of the co-efficients

$$X', \; Y', \; \frac{Z'}{2}, \; \frac{V'}{2 \, . \, 3}.$$

Now, it follows from the preceding law for derived polynomials, that

$X' = (3)^4 - 12 \, . \, (3)^3 + 17 \, . \, (3)^2 - 9 \, . \, (3)^1 + 7$, or $X' = -110$;

$Y' = 4 \, . \, (3)^3 - 36 \, . \, (3)^2 + 34 \, . \, (3)^1 - 9$, or - - - $Y' = -123$;

$\dfrac{Z'}{2} = 6 \, . \, (3)^2 - 36 \, . \, (3)^1 + 17$, or - - - - - $\dfrac{Z'}{2} = -37$;

$\dfrac{V'}{2 \, . \, 3} = 4 \, . \, (3)^1 - 12$ - - - - - - - - - $\dfrac{V'}{2 \, . \, 3} = 0.$

Therefore the transformed equation becomes

$$u^4 - 37u^2 - 123u - 110 = 0.$$

3. Transform the equation

$$4x^3 - 5x^2 + 7x - 9 = 0$$

into another equation, the roots of which shall exceed those of the given equation by unity.

Make, $x = u - 1$; whence $x' = -1$;

and the transformed equation will be of the form

$$X' + Y'u + \frac{Z'}{1.2}u^2 + \frac{V'}{1.2.3}u^3.$$

Hence, we have

$$X' = 4.(-1)^3 - 5.(-1)^2 + 7.(-1)^1 - 9, \text{ or } X' = -25;$$

$$Y' = 12.(-1)^2 - 10.(-1)^1 + 7 \quad \text{- - - - -} \quad Y' = +29;$$

$$\frac{Z'}{2} = 12.(-1)^1 - 5 \quad \text{- - - - - - - -} \quad \frac{Z'}{2} = -17;$$

$$\frac{V'}{2.3} = 4 \quad \text{- - - - - - - - - - - -} \quad \frac{V'}{2.3} = +4.$$

Therefore, the transformed equation becomes

$$4u^3 - 17u^2 + 29u - 25 = 0.$$

4. What is the transformed equation, if the second term be made to disappear in the equation

$$x^5 - 10x^4 + 7x^3 + 4x - 9 = 0?$$

Ans. $u^5 - 33u^3 - 118u^2 - 152u - 73 = 0.$

5. What is the transformed equation, if the second term be made to disappear in the equation

$$3x^3 + 15x^2 + 25x - 3 = 0?$$

Ans. $3u^3 - \frac{152}{9} = 0.$

6. Transform the equation

$$3x^4 - 13x^3 + 7x^2 - 8x - 9 = 0$$

into another, the roots of which shall be less than the roots of the given equation by $\frac{1}{3}$.

Ans. $3u^4 - 9u^3 - 4u^2 - \frac{65}{9}u - \frac{102}{9} = 0.$

Properties of derived Polynomials.

299. We will now develop some of the remarkable properties of derived polynomials.

Let $X = x^m + Px^{m-1} + Qx^{m-2} \ldots Tx + U = 0$

be a given equation, and a, b, c, d, &c., its m roots. We shall then have (Art. 281),

$$x^m + Px^{m-1} + Qx^{m-2} \ldots = (x-a)(x-b)(x-c)\ldots(x-l).$$

Making $x = x' + u,$

or omitting the accents, and substituting $x+u$ for x, and we have

$$(x+u)^m + P(x+u)^{m-1} + \ldots = (x+u-a)(x+u-b)\ldots;$$

or, changing the order of x and u, in the second member, and regarding $x-a$, $x-b$, ... each as a single quantity,

$$(x+u)^m + P(x+u)^{m-1} \ldots = (u+\overline{x-a})(u+\overline{x-b})\ldots(u+\overline{x-l}).$$

Now, by performing the operations indicated in the two members, we shall, by the preceding Article, obtain for the first member,

$$X + Yu + \frac{Z}{2}u^2 + \ldots u^m;$$

X being the first member of the proposed equation, and Y, Z ... the derived polynomials of this member.

With respect to the second member, it follows from Art. 295,

1st. That the part involving u^0, or the last term, is equal to the product $(x-a)(x-b)\ldots(x-l)$ of the factors of the proposed equation.

2d. The co-efficient of u is equal to the sum of the products of these m factors, taken $m-1$ and $m-1$.

3d. The co-efficient of u^2 is equal to the sum of the products of these m factors, taken $m-2$ and $m-2$; and so on.

Moreover, since the two members of the last equation are identical, the co-efficients of the same powers are equal. Hence,

$$X = (x-a)(x-b)(x-c)\ldots(x-l),$$

which was already known. Hence also, Y, or the first-derived polynomial, *is equal to the sum of the products of the* m *factors of the first degree in the proposed equation, taken* m − 1 *and* m − 1; *or equal to the sum of all the quotients that can be obtained by*

dividing X *by each of the* m *factors of the first degree in the proposed equation; that is,*

$$Y = \frac{X}{x-a} + \frac{X}{x-b} + \frac{X}{x-c} + \dots \frac{X}{x-l}.$$

Also, $\frac{Z}{2}$, that is, the second-derived polynomial, divided by 2, is equal to the sum of the products of the m factors of the proposed equation taken $m-2$ and $m-2$, or equal to the sum of the quotients that can be obtained by dividing X by each of the factors of the second degree; that is,

$$\frac{Z}{2} = \frac{X}{(x-a)(x-b)} + \frac{X}{(x-a)(x-c)} \dots \frac{X}{(x-k)(x-l)};$$

and so on.

Of equal Roots.

300. An equation is said to contain equal roots, when its first member contains equal factors. When this is the case, the derived polynomial, which is the sum of the products of the m factors taken $m-1$ and $m-1$, contains a factor in its different parts, which is two or more times a factor of the proposed equation (Art. 299).

Hence, *there must be a common divisor between the first member of the proposed equation, and its first-derived polynomial.*

It remains to ascertain the relation between this common divisor and the equal factors.

301. *Having given an equation, it is required to discover whether it has equal roots, and to determine these roots if possible.*

Let us make

$$X = x^m + Px^{m-1} + Qx^{m-2} + \dots + Tx + U = 0,$$

and suppose that the second member contains n factors equal to $x-a$, n' factors equal to $x-b$, n'' factors equal to $x-c\dots$, and also, the simple factors $x-p$, $x-q$, $x-r\dots$; we shall then have,

$$X = (x-a)^n (x-b)^{n'} (x-c)^{n''} \dots (x-p)(x-q)(x-r) \quad (1).$$

We have seen that Y, or the derived polynomial of X, is *the sum of the quotients obtained by dividing* X *by each of the* m *factors of the first degree in the proposed equation* (Art. 299).

Now, since X contains n factors equal to $x - a$, we shall have n partial quotients equal to $\frac{X}{x-a}$; and the same reasoning applies to each of the repeated factors, $x - b$, $x - c \dots$ Moreover, we can form but one quotient for each simple factor, which is of the form,

$$\frac{X}{x-p}, \quad \frac{X}{x-q}, \quad \frac{X}{x-r} \dots$$

Therefore, the first-derived polynomial is of the form,

$$Y = \frac{nX}{x-a} + \frac{n'X}{x-b} + \frac{n''X}{x-c} + \dots + \frac{X}{x-p} + \frac{X}{x-q} + \frac{X}{x-r} + \dots \quad (2).$$

By examining the form of the value of X in equation (1), it is plain that

$$(x-a)^{n-1}, \quad (x-b)^{n'-1}, \quad (x-c)^{n''-1} \dots$$

are factors common to all the terms of the polynomial; hence the product

$$(x-a)^{n-1} \times (x-b)^{n'-1} \times (x-c)^{n''-1} \dots$$

is a common divisor of Y. Moreover, it is evident that it will also divide X: it is therefore a common divisor of X and Y; and it is their greatest common divisor.

For, the prime factors of X are $x - a$, $x - b$, $x - c \dots$, and $x - p$, $x - q$, $x - r \dots$; now, $x - p$, $x - q$, $x - r$, cannot divide Y, since some one of them will be wanting in each of the parts of Y, while it will be a factor of all the other parts.

Hence, the greatest common divisor of X and Y is

$$D = (x-a)^{n-1}(x-b)^{n'-1}(x-c)^{n''-1} \dots; \text{ that is,}$$

The greatest common divisor is composed of the product of those factors which enter two or more times in the given equation, each raised to a power less by unity than in the primitive equation.

302. From the above we deduce the following method for finding the equal roots.

To discover whether an equation

$$X = 0$$

contains any equal roots, *form Y or the derived polynomial of X; then seek for the greatest common divisor between X and Y*; if one cannot be obtained, the equation has no equal roots, or equal factors.

If we find a common divisor D, and it is of the first degree, or of the form $x - h$, make $x - h = 0$, whence $x = h$. *We then conclude, that the equation has two roots equal to* h, *and has but one species of equal roots,* from which it may be freed by dividing X by $(x - h)^2$.

If D is of the second degree with reference to x, *resolve the equation* $D = 0$. There may be two cases; the two roots will be equal, or they will be unequal.

1st. When we find $D = (x - h)^2$, *the equation has three roots equal to* h, *and has but one species of equal roots,* from which it can be freed by dividing X by $(x - h)^3$.

2d. When D is of the form $(x - h)(x - h')$, the proposed equation *has two roots equal to h,* and *two equal to h',* from which it may be freed by dividing X by $(x - h)^2(x - h')^2$, or by D^2.

Suppose now that D is of any degree whatever; *it is necessary,* in order to know the species of equal roots, and the number of roots of each species, *to resolve completely the equation*

$$D = 0.$$

Then, every simple root of D will be twice a root of the given equation; every double root of D will be three times a root of the given equation; and so on.

As to the simple roots of

$$X = 0,$$

we begin by freeing this equation of the equal factors contained in it, and the resulting equation, $X' = 0$, will make known the simple roots.

EXAMPLES.

1. Determine whether the equation

$$2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0$$

contains equal roots.

We have for the first-derived polynomial (Art. 297),

$$8x^3 - 36x^2 + 38x - 6.$$

Now, seeking for the greatest common divisor of these polynomials, we find

$$D = x - 3 = 0, \quad \text{whence} \quad x = 3;$$

hence, the given equation has *two* roots equal to 3.

Dividing its first member by $(x-3)^2$, we obtain

$$2x^2+1=0; \quad \text{whence} \quad x=\pm\frac{1}{2}\sqrt{-2}.$$

The equation, therefore, is completely resolved, and its roots are

$$3, \quad 3, \quad +\frac{1}{2}\sqrt{-2} \quad \text{and} \quad -\frac{1}{2}\sqrt{-2}.$$

2. For a second example, take

$$x^5-2x^4+3x^3-7x^2+8x-3=0.$$

The first-derived polynomial is

$$5x^4-8x^3+9x^2-14x+8;$$

and the common divisor,

$$x^2-2x+1=(x-1)^2:$$

hence, the proposed equation has *three* roots equal to 1.

Dividing its first member by

$$(x-1)^3=x^3-3x^2+3x-1,$$

the quotient is

$$x^2+x+3=0; \quad \text{whence} \quad x=\frac{-1\pm\sqrt{-11}}{2};$$

thus, the equation is completely resolved.

3. For a third example, take the equation,

$$x^7+5x^6+6x^5-6x^4-15x^3-3x^2+8x+4=0.$$

The first-derived polynomial is

$$7x^6+30x^5+30x^4-24x^3-45x^2-6x+8;$$

and the common divisor is

$$x^4+3x^3+x^2-3x-2.$$

The equation

$$x^4+3x^3+x^2-3x-2=0$$

cannot be resolved directly, but by applying the method of equal roots to it, that is, by seeking for a common divisor between its first member and its derived polynomial

$$4x^3+9x^2+2x-3,$$

we find a common divisor, $x+1$; which proves that the *square* of $x+1$ is a factor of $x^4+3x^3+x^2-3x-2$, and the *cube* of $x+1$, a factor of the first member of the given equation.

Dividing

$$x^4 + 3x^3 + x^2 - 3x - 2 \quad \text{by} \quad (x+1)^2 = x^2 + 2x + 1,$$

we have $x^2 + x - 2$, which being placed equal to zero, gives the two roots $x = 1$, $x = -2$, or the two factors, $x - 1$ and $x + 2$. Hence we have

$$x^4 + 3x^3 + x^2 - 3x - 2 = (x+1)^2 (x-1)(x+2).$$

Therefore, the first member of the proposed equation is equal to

$$(x+1)^3 (x-1)^2 (x+2)^2;$$

that is, the proposed equation has *three* roots equal to -1, *two* equal to $+1$, and *two* equal to -2.

4. What are the equal factors of the equation

$$x^7 - 7x^6 + 10x^5 + 22x^4 - 43x^3 - 35x^2 + 48x + 36 = 0.$$

Ans. $(x-2)^2 (x-3)^2 (x+1)^3 = 0.$

5. What are the equal factors in the equation

$$x^7 - 3x^6 + 9x^5 - 19x^4 + 27x^3 - 33x^2 + 27x - 9 = 0.$$

Ans. $(x-1)^3 (x^2+3)^2 = 0.$

Elimination.

303. To eliminate between two equations of any degree whatever, involving two unknown quantities, is to obtain, by a series of operations, performed on these equations, *a single equation which contains but one of the unknown quantities,* and which gives all the values of this unknown quantity that will, taken in connexion with the corresponding values of the other unknown quantity, satisfy at the same time both the given equations.

This new equation, *which is a function of one of the unknown quantities,* is called *the final equation,* and the values of the unknown quantity found from it, are called *compatible values.*

Elimination by Means of Indeterminate Multipliers.

304. Let there be the equations

$$ax + by - c = 0,$$
$$a'x + b'y - c' = 0.$$

If we multiply the first by m, and subtract the second from the product, we have

$$(ma - a')x + (mb - b')y - mc + c' = 0 \ldots (1).$$

Now, since the value of m is entirely arbitrary, we may give it such a value as to render the co-efficient of x zero, which gives

$$ma - a' = 0, \quad \text{whence} \quad m = +\frac{a'}{a},$$

and $$(mb - b')y - mc + c' = 0 \ldots\ldots (2).$$

Substituting in equation (2) the value of m, and we have

$$y = \frac{\frac{a'}{a}.c - c'}{\frac{a'}{a}.b - b'} = \frac{a'c - ac'}{a'b - ab'} = \frac{ac' - a'c}{ab' - a'b}.$$

Had we chosen to attribute to m such a value as to render the co-efficient of y zero in equation (1), we should have had

$$mb - b' = 0, \quad \text{whence} \quad m = \frac{b'}{b}$$

and $$(ma - a')x - mc + c' = 0 \ldots\ldots (3).$$

Substituting in equation (3) the value of m, we obtain

$$x = \frac{\frac{b'}{b}.c - c'}{\frac{b'}{b}.a - a'} = \frac{b'c - bc'}{ab' - a'b}.$$

The above values for x and y are the same as those determined in Art. 97. The principle explained above is applicable to three or more equations, involving a like number of unknown quantities.

305. Of all the known methods of elimination, however, *the method of the common divisor* is, in general, the best; it is this method which we are going to develop.

Let $f(x, y) = 0 = A$, and $f'(x, y) = 0 = B$,
be any two equations whatever, in which f and f' denote any functions of x and y.

Suppose the final equation involving y obtained, and let us try to discover some property of the roots of this equation, which may serve to determine it.

Let $$y = a$$
be one of the values of y which will satisfy both the given equations. This is called a *compatible* value of y. It is plain, that, since this value of y, in connexion with a certain value of x, will satisfy both equations, that if it be substituted in them, there will result two

equations involving x alone, *which will admit of at least one common value of x;* and to this common value there will correspond a common divisor involving x (Art. 279). This common divisor will be of the first, or of a higher degree with respect to x, according as the particular value of $y = a$ corresponds to one or more values of x.

Reciprocally, *every value of y which,* substituted in the two equations, gives a common divisor involving x, *is necessarily a compatible value,* because it then evidently satisfies the two equations at the same time with the value or values of x found from this common divisor when put equal to 0.

306. We will remark, that, before *the substitution, the first members of the equations cannot,* in general, *have a common divisor* which is a function of one or both of the unknown quantities.

For, let us suppose for a moment that the equations

$$A = 0, \quad B = 0,$$

are of the form

$$A' \times D = 0, \quad B' \times D = 0.$$

D being a function of x and y.

Making separately $D = 0$, we obtain a single equation involving two unknown quantities, which can be satisfied with an *infinite number of systems of values.* Moreover, every system which renders D equal to 0, would at the same time cause $A'D$, $B'D$ to vanish, and would consequently satisfy the equations

$$A = 0 \quad \text{and} \quad B = 0.$$

Thus, the hypothesis of a common divisor of the two polynomials A and B, containing x and y, would bring with it as a consequence that the proposed equations were indeterminate. Therefore, if there exists a common divisor, involving x and y, of the two polynomials A and B, the proposed equations will be *indeterminate,* that is, they may be satisfied by an infinite number of systems of values of x and y. Then there would be no data to determine a *final equation* in y, since the number of values of y is *infinite.*

If the two polynomials A and B were of the form

$$A' \times D, \quad B' \times D,$$

D being a function of x only, we might conceive the equation $D = 0$ resolved with reference to x, which would give one or

more values for this unknown. Each of these values substituted in the equations

$$A' \times D = 0 \quad \text{and} \quad B' \times D = 0,$$

would verify them, without regard to the value of y, since D must be nothing, in consequence of the substitution of the value of x. Therefore, in this case, the proposed equations would admit of a *finite number of values* for x, but of an infinite number of values for y, and then there could not exist a final equation in y.

Hence, when the equations

$$A = 0, \quad B = 0,$$

are determinate, that is, when they admit only of a *limited number* of systems of values for x and y, their first members cannot have *for a common divisor a function of these unknown quantities*, unless a particular substitution has been made for one of these quantities.

307. From this it is easy to deduce a process for obtaining the *final equation* involving y.

Since the characteristic property of every compatible value of y is, that being substituted in the first members of the two equations, it gives them a common divisor involving x, which they had not before, it follows, that if to the two proposed polynomials, arranged with reference to x, we apply the process for finding the greatest common divisor, we shall generally not find one. But, by continuing the operation properly, we shall arrive at a remainder independent of x, but which is a function of y, and which, placed equal to 0, will give the required *final equation*. For, every value of y found from this equation, reduces to nothing the last remainder of the operation for finding the common divisor; it is, then, such, that substituted in the preceding remainder, it will render this remainder a common divisor of the first members A and B. Therefore, each of the roots of the equation thus formed, is a compatible value of y.

308. Admitting that the final equation may be completely resolved, which would give all the compatible values, it would afterward be necessary to obtain the corresponding values of x. Now, it is evident that it would be sufficient for this, to substitute the different values of y in the remainder preceding the last, put the polynomial involving x which results from it, equal to 0, and find

from it the values of x; for these polynomials are nothing more than the divisors involving x, which become common to A and B.

But as the final equation is generally of a degree superior to the second, we cannot here explain the methods of finding the values of y. Indeed, our design was principally to show that, *two equations of any degree being given, we can, without supposing the resolution of any equation, arrive at another equation, containing only one of the unknown quantities which enter into the proposed equation.*

EXAMPLES.

1. Having given the equations

$$\begin{aligned} A &= x^2 + xy + y^2 - 1 = 0, \\ B &= x^3 + y^3 = 0, \end{aligned}$$

to find the final equation in y.

First Operation.

$$\begin{array}{l|l} x^3 + y^3 & x^2 + xy + y^2 - 1 \\ \cline{2-2} x^3 + yx^2 + (y^2 - 1)x & x - y = Q \\ \cline{1-1} -yx^2 - (y^2 - 1)x + y^3 & \\ -yx^2 - y^2x - y^3 + y & \\ \cline{1-1} \end{array}$$

$$R = x + 2y^3 - y = \text{1st remainder.}$$

Second Operation.

$$\begin{array}{l|l} x^2 + yx + y^2 - 1 & x + 2y^3 - y \\ \cline{2-2} x^2 + (2y^3 - y)x & x - (2y^3 - 2y) \\ \cline{1-1} -(2y^3 - 2y)x + y^2 - 1 & \\ -(2y^3 - 2y)x - 4y^6 + 6y^4 - 2y^2 & \\ \cline{1-1} \end{array}$$

$$R' = 4y^6 - 6y^4 + 3y^2 - 1 = \text{2d remainder.}$$

Hence, the final equation in y, is

$$4y^6 - 6y^4 + 3y^2 - 1 = 0.$$

If it were required to find the final equation in x, we observe that x and y enter in the same manner into the original equations; hence, x may be changed into y and y into x, without destroying the equality of the members. Therefore,

$$4x^6 - 6x^4 + 3x^2 - 1 = 0$$

is the final equation in x.

2. Find the final equation in y, from the equations

$$A = x^3 - 3yx^2 + (3y^2 - y + 1)x - y^3 + y^2 - 2y = 0,$$
$$B = x^2 - 2yx + y^2 - y = 0.$$

First Operation.

$$\begin{array}{l|l}
x^3 - 3yx^2 + (3y^2 - y + 1)x - y^3 + y^2 - 2y & x^2 - 2xy + y^2 - y \\
\underline{x^3 - 2yx^2 + (y^2 - y)x} & x - y = Q \\
\quad - yx^2 + (2y^2 + 1)x - y^3 + y^2 - 2y & \\
\quad - yx^2 + 2y^2x \qquad \underline{- y^3 + y^2} & \\
\qquad\qquad x - 2y = R. &
\end{array}$$

Second Operation.

$$\begin{array}{l|l}
x^2 - 2xy + y^2 - y & x - 2y \\
\underline{x^2 - 2xy} & x = Q' \\
\qquad y^2 - y = R'. &
\end{array}$$

Hence, $$y^2 - y = 0$$

is the final equation in y. This equation gives

$$y = 1 \quad \text{and} \quad y = 0.$$

Placing the preceding remainder equal to zero (Art. 308), and substituting therein the values of

$$y = 1 \quad \text{and} \quad y = 0,$$

we find for the corresponding values of x,

$$x = 2 \quad \text{and} \quad x = 0;$$

from which the given equations may be entirely resolved.

CHAPTER XI.

RESOLUTION OF NUMERICAL EQUATIONS INVOLVING ONE OR MORE UNKNOWN QUANTITIES.

309. THE principles established in the preceding chapter, are applicable to all equations, whether their co-efficients are numerical or algebraic, and these principles are the elements which are to be employed in the resolution of equations of the higher degrees.

It has been already remarked, that analysts have hitherto been able to resolve only the general equations of the third and fourth degrees. The general formulas which have been obtained for the resolution of algebraic equations of the higher degrees, are so complicated and inconvenient, even when they can be applied, that the problem of the resolution of algebraic equations, of any degree whatever, may be regarded as more curious than useful.

Therefore, analysts have principally directed their researches to the resolution of *numerical equations*, that is, to those which arise from the algebraic translation of a problem in which the given quantities are particular numbers. Methods have been found, by means of which, the roots of a *numerical equation of any given degree*, may always be determined.

It is proposed to develop these methods in this chapter.

To render the reasoning general, we will represent the proposed equation by

$$X = x^m + Px^{m-1} + Qx^{m-2} + \ldots = 0.$$

in which P, Q . . . denote particular numbers which are real, and either positive or negative.

First Principle.

310. If we substitute for x a number a, and denote by A what X becomes under this supposition; and again substitute $a+u$ for x, and denote the new polynomial by A': then, *u may be taken so small, that the difference between A' and A shall be less than any assignable quantity.*

If now, we denote by B, C, D, ... , what the co-efficients Y, $\frac{Z}{2}$, $\frac{V}{2.3}$ (Art. 297), become, when we make $x=a$, we shall have for the polynomial X, under the supposition that $a+u$ is substituted for x,

$$A + Bu + Cu^2 + Du^3 + \ldots u^n,$$

equal to $\quad A + u(B + Cu + Du^2 + \ldots u^{n-1}) = A'.$

Now, the quantity

$$u(B + Cu + Du^2 + \ldots u^{n-1})$$

is the difference between A' and A; and it is required to show that this difference may be rendered less than any assignable quantity, by attributing a value sufficiently small to u.

Let us take the most unfavorable case that can occur, viz., let us suppose that every co-efficient is positive, and that each is equal to the largest, which we will designate by K. Then,

$$Ku(1 + u + u^2 + \ldots u^{m-1}) = u(A + Bu + \ldots + u^{m-1});$$

and in any other case,

$$Ku(1 + u + u^2 + \ldots u^{m-1}) > u(A + Bu + \ldots u^{m-1}).$$

But we have, by Art. 61,

$$Ku(1 + u + u^2 + \ldots u^{m-1}) = Ku\left(\frac{1-u^m}{1-u}\right);$$

and $$\frac{Ku}{1-u}(1-u^m) < \frac{Ku}{1-u};$$

when $$u < 1.$$

This being premised, if we wish the difference between A' and A to be less than any number N, let us make u such, that

$$\frac{Ku}{1-u} = \text{ or } < N, \text{ which requires that, } u = \text{ or } < \frac{N}{N+K},$$

and any value of u which will fulfil this last condition, will satisfy the inequality

$$Ku(1 + u + u^2 + \ldots u^{m-1}) < N,$$

and consequently, render

$$u(B + Cu + Du^2 + \ldots u^{m-1}) < N;$$

in which the inequality is greater even than in the expression above.

Second Principle.

311. *If two numbers* p *and* q, *substituted in succession in the place of* x *in a numerical equation, give two results affected with contrary signs, the proposed equation contains a real root, comprehended between these two numbers.*

Let us suppose that p, when substituted for x in the equation

$$X = 0, \quad \text{gives} \quad +R,$$

and that q substituted in the equation

$$X = 0, \quad \text{gives} \quad -R'.$$

Let us now suppose x to vary between the values of p and q by so small a quantity, that the difference between any two corresponding consecutive values of X shall be less than any assignable quantity; in which case, we say that X is subject to the *law of continuity*, or that it passes through all the intermediate values between R and $-R'$.

Now, a quantity which is constantly finite, and subject to the law of continuity, cannot change its sign from positive to negative, or from negative to positive, without passing through zero: hence, there is at least one number between p and q which will satisfy the equation

$$X = 0,$$

and consequently, one root of the equation lies between these numbers.

312. We have shown in the last article, that if two numbers be substituted, in succession, for the unknown quantity in any equation, and give results affected with contrary signs, that there will be at least one real root comprehended between them. We are not, however, to conclude that there may not be more than one; nor

that the substitution, in succession, of two numbers which include roots of the equation, will *necessarily* give results affected with contrary signs.

Third Principle.

313. *When an uneven number of the real roots of an equation are comprehended between two numbers, the results obtained by substituting these numbers in succession for* x, *will be affected with contrary signs; but if they comprehend an even number of roots, the results obtained by their substitution will be affected with the same sign.*

To make this proposition as clear as possible, denote by a, b, c, ... those roots of the proposed equation,

$$X = 0,$$

which are supposed to be comprehended between p and q, and by Y, the product of the factors of the first degree, with reference to x, corresponding to the remaining real roots and to the imaginary roots of the given equation.

The first member, X, can then be put under the form

$$(x - a)(x - b)(x - c) \ldots \times Y = X.$$

Now, substituting p and q in place of x, we shall obtain the two results

$$(p - a)(p - b)(p - c) \ldots \times Y',$$
$$(q - a)(q - b)(q - c) \ldots \times Y'',$$

Y' and Y'' representing what Y becomes, when we replace in succession, x by p and q. These two quantities Y' and Y'', are affected with the same sign; for, if they were not, by the second principle there would be at least one real root comprised between p and q, which is contrary to the hypothesis.

To determine the signs of the above results more easily, divide the first by the second, and we obtain

$$\frac{(p - a)(p - b)(p - c) \ldots \times Y'}{(q - a)(q - b)(q - c) \ldots \times Y''},$$

which can be written thus,

$$\frac{p - a}{q - a} \times \frac{p - b}{q - b} \times \frac{p - c}{q - c} \times \ldots \frac{Y'}{Y''}.$$

Now, since the roots $a, b, c, \ldots$ are comprised between p and q, we have

$$p \gtrless a, b, c, d \ldots,$$

and
$$q \lessgtr a, b, c, d \ldots;$$

whence we deduce

$$p - a, \ p - b, \ p - c, \ldots \gtrless 0,$$

and
$$q - a, \ q - b, \ q - c, \ldots \lessgtr 0.$$

Hence, since $p - a$ and $q - a$ are affected with contrary signs, as well as $p - b$ and $q - b$, $p - c$ and $q - c \ldots$, the partial quotients

$$\frac{p-a}{q-a}, \ \frac{p-b}{q-b}, \ \frac{p-c}{q-c}, \ \text{\&c.},$$

are all *negative*. Moreover, $\frac{Y'}{Y''}$ is essentially positive, since Y' and Y'' are affected with the same sign; therefore, the product

$$\frac{p-a}{q-a} \times \frac{p-b}{q-b} \times \frac{p-c}{q-c} \times \ldots \frac{Y'}{Y''},$$

will be *negative*, when the number of roots, $a, b, c \ldots$, comprehended between p and q, is uneven, and *positive* when the number is even.

Consequently, the two results

$$(p - a)(p - b)(p - c) \ldots \times Y',$$

and
$$(q - a)(q - b)(q - c) \ldots \times Y'',$$

will have contrary or the same signs, according as the number of roots comprised between p and q is *uneven* or *even*.

Limits of Real Roots.

314. The different methods for resolving numerical equations, consist, generally, in substituting particular numbers in the proposed equation, in order to discover if these numbers verify it, or whether there are roots comprised between them. But by reflect-

ing a little on the composition of the first member of the general equation

$$x^m + Px^{m-1} + Qx^{m-2} \dots + Tx + U = 0,$$

we become sensible, that there are certain numbers, above which it would be useless to substitute, because all of them above a certain limit, would give positive results.

315. Let it now be required to resolve the following question: *To determine a number, which substituted in place of* x *will render the first term* x^m *greater than the arithmetical sum of all the other terms.*

Suppose all the terms of the equation to be negative, except the first, so that

$$x^m - Px^{m-1} - Qx^{m-2} \dots - Tx - U = 0.$$

It is required to find a number for x which will render

$$x^m > Px^{m-1} + Qx^{m-2} + \dots + Tx + U.$$

Let k denote the greatest co-efficient, and substitute it in place of the co-efficients; the inequality will then become

$$x^m > kx^{m-1} + kx^{m-2} + \dots + kx + k.$$

It is evident that every number substituted for x which will satisfy this condition, will for a stronger reason, satisfy the preceding. Now, dividing this inequality by x^m, it becomes

$$1 > \frac{k}{x} + \frac{k}{x^2} + \frac{k}{x^3} + \dots + \frac{k}{x^{m-1}} + \frac{k}{x^m}.$$

Making $x = k$, the second member becomes $\frac{k}{k} = 1$ plus a series of positive fractions. The number k will therefore not satisfy the inequality; but by supposing $x = k + 1$, we obtain for the second member the series of fractions

$$\frac{k}{k+1} + \frac{k}{(k+1)^2} + \frac{k}{(k+1)^3} + \dots + \frac{k}{(k+1)^{m-1}} + \frac{k}{(k+1)^m},$$

which, considered in an inverse order, is an increasing geometrical progression, the first term of which is $\frac{k}{(k+1)^m}$, the ratio $k + 1$, and the last term $\frac{k}{k+1}$; hence, the expression for the sum of all the terms is (Art. 192),

$$\frac{\dfrac{k}{k+1}\cdot(k+1) - \dfrac{k}{(k+1)^m}}{k+1-1} = 1 - \frac{1}{(k+1)^m},$$

which is evidently less than unity.

Now, any number $> (k+1)$, put in place of x, will render the sum of the fractions $\frac{k}{x} + \frac{k}{x^2} + \dots$ still less. Therefore,

The greatest co-efficient plus unity, or any greater number, being substituted for x, *will render the first term* x^m *greater than the arithmetical sum of all the other terms.*

316. Every number which exceeds the greatest of the positive roots of an equation, *is called a superior limit of the positive roots.*

From this definition, it follows, that this limit is susceptible of an infinite number of values. For, when a number is found to exceed the greatest positive root, every number greater than this, is also a superior limit.

But since the largest of the positive roots will, when substituted for x, merely reduce the first member to zero, it follows, that we shall be sure of obtaining a superior limit of the positive roots by finding *a number, which substituted in place of* x *renders the first member positive, and which at the same time is such, that every greater number will also give a positive result.*

Hence, *the greatest co-efficient of* x *plus unity, is a superior limit of the positive roots.*

Ordinary Limit of the Positive Roots.

317. The limit of the positive roots obtained in the last article, is commonly much too great, because, in general, the equation contains several positive terms. We will, therefore, seek for a limit suitable for all equations.

Let x^{m-n} denote that power of x, corresponding to the first negative term which follows x^m, and let us consider the most unfavorable case, viz., that in which all the succeeding terms are negative and affected with the greatest of the negative co-efficients in the equation.

Let S denote this co-efficient. What conditions will render

$$x^m > Sx^{m-n} + Sx^{m-n-1} + \dots Sx + S\,?$$

Dividing both members of this inequality by x^m, we have

$$1 > \frac{S}{x^n} + \frac{S}{x^{n+1}} + \frac{S}{x^{n+2}} + \ldots + \frac{S}{x^{m-1}} + \frac{S}{x^m}.$$

Now, by supposing

$x = \sqrt[n]{S} + 1$, or for simplicity, making $\sqrt[n]{S} = S'$

which gives, $S = S'^n$, and $x = S' + 1$,

the second member of the inequality will become,

$$\frac{S'^n}{(S'+1)^n} + \frac{S'^n}{(S'+1)^{n+1}} + \ldots + \frac{S'^n}{(S'+1)^{m-1}} + \frac{S'^n}{(S'+1)^m},$$

which is a progression by quotients, of which $\frac{S'^n}{(S'+1)^m}$ is the first term, $S'+1$ the ratio, and $\frac{S'^n}{(S'+1)^n}$ the last term. Hence, the expression for the sum of all the terms is (Art. 192),

$$\frac{\frac{S'^n}{(S'+1)^n}\cdot(S'+1) - \frac{S'^n}{(S'+1)^m}}{S'+1-1} = \frac{S'^{n-1}}{(S'+1)^{n-1}} - \frac{S'^{n-1}}{(S'+1)^m},$$

which is evidently less than 1.

Moreover, every number $> S'+1$ or $\sqrt[n]{S}+1$, will, when substituted for x, render the sum of the fractions

$$\frac{S}{x^n} + \frac{S}{x^{n+1}} + \ldots$$

still smaller, since the numerators remaining the same, the denominators will increase.

Hence, $\sqrt[n]{S} + 1$, and any greater number, will render the first term x^m greater than the arithmetical sum of all the negative terms of the equation, and will consequently give a positive result for the first member. Therefore,

Unity increased by that root of the greatest negative co-efficient whose index is the number of terms which precede the first negative term, is a superior limit of the positive roots of the equation. If the co-efficient of a term is 0, the term must still be counted.

Make $n = 1$, in which case the first negative term is the second term of the equation; the limit becomes

$$\sqrt[1]{S} + 1 = S + 1;$$

that is, *the greatest negative co-efficient plus unity.*

Let $n = 2$; then, the limit is $\sqrt[2]{S} + 1$. When $n = 3$, the limit is $\sqrt[3]{S} + 1$.

EXAMPLES.

1. What is the superior limit of the positive roots in the equation

$$x^4 - 5x^3 + 37x^2 - 3x + 39 = 0.$$

Ans. $\sqrt[n]{S} + 1 = \sqrt[1]{5} + 1 = 6.$

2 What is the superior limit of the positive roots in the equation

$$x^5 + 7x^4 - 12x^3 - 49x^2 + 52x - 13 = 0.$$

Ans. $\sqrt[n]{S} + 1 = \sqrt{49} + 1 = 8.$

3. What is the superior limit of the positive roots in the equation

$$x^4 + 11x^2 - 25x - 67 = 0.$$

In this example, we see that the second term is wanting, that is, its co-efficient is zero; but the term must still be counted in fixing the value of n. We also see, that the largest negative co-efficient of x is found in the last term where the exponent of x is zero. Hence,

$$\sqrt[n]{S} + 1 = \sqrt[3]{67} + 1;$$

and therefore, 6 is the least whole number that will certainly fulfil the conditions.

Smallest Limit in Entire Numbers.

318. In Art. 316, it was shown that the greatest co-efficient of x plus unity, is a superior limit of the positive roots. In the last article we found a limit still less; and we now propose to find the smallest limit in whole numbers.

Let $$X = 0;$$

be the proposed equation. If in this equation we make $x = x' + u$, x' being indeterminate, we shall obtain (Art. 297),

$$X' + Y'u + \frac{Z'}{2}u^2 + \ldots + u^m = 0 \quad (1).$$

Let us suppose, that after successive trials we have determined a number for x', which substituted in

$$X', \ Y', \ \frac{Z'}{2} \ldots,$$

renders all these co-efficients positive at the same time; this number will be greater than the greatest positive root of the equation

$$X = 0.$$

For, if the co-efficients of equation (1) are all positive, no positive number can verify it; therefore, *all* of the real values of u must be *negative*. But from the equation

$$x = x' + u, \quad \text{we have} \quad u = x - x';$$

and in order that every value of u, corresponding to each of the values of x and x', may be negative, it is necessary that the greatest positive value of x should be less than the value of x'.

EXAMPLES

Let $$x^4 - 5x^3 - 6x^2 - 19x + 7 = 0.$$

As x' is indeterminate, we may, to avoid the inconvenience of writing the primes, retain the letter x in the formation of the derived polynomials; and we have

$$X = x^4 - 5x^3 - 6x^2 + 19x + 7,$$

$$Y = 4x^3 - 15x^2 - 12x - 19,$$

$$\frac{Z}{2} = 6x^2 - 15x - 6,$$

$$\frac{V}{2.3} = 4x - 5.$$

The question is now reduced to finding the smallest entire number which, substituted in place of x, will render all of these polynomials positive.

It is plain that 2 and every number > 2, will render the polynomial of the first degree positive.

But 2, substituted in the polynomial of the second degree, gives a negative result; and 3, or any number > 3, gives a positive result.

Now 3 and 4, substituted in succession in the polynomial of the third degree, give negative results; but 5, and any greater number, gives a positive result.

Lastly, 5 substituted in X, gives a negative result, and so does 6; for the first three terms, $x^4 - 5x^3 - 6x^2$, are equivalent to the expression $x^3(x - 5) - 6x^2$, which reduces to 0 when $x = 6$; but $x = 7$ evidently gives a positive result. Hence 7, which here

stands for x', *is a superior limit of the positive roots of the given equation.* Since it has been shown that 6 gives a negative result, it follows that there is at least one real root between 6 and 7.

2. Applying this method to the equation

$$x^5 - 3x^4 - 8x^3 - 25x^2 + 4x - 39 = 0,$$

the superior limit is found to be 6.

3. We find 7 to be the superior limit of the positive roots of the equation

$$x^5 - 5x^4 - 13x^3 + 17x^2 - 69 = 0.$$

This method is seldom used, except in finding incommensurable roots.

Superior Limit of negative Roots.—Inferior Limit of positive and negative Roots.

319. Having found the superior limit of the positive roots, it only remains to find the inferior limit, and the superior and inferior limits of the negative roots.

Let, L = superior limit of positive roots.
L' = inferior limit of positive roots.
L'' = superior limit (that is, numerically) of negative roots.
L''' = inferior limit of negative roots.

1st. If in any equation $X = 0$, we make $x = \frac{1}{y}$, we have a derived equation $Y = 0$. We know from the relation $x = \frac{1}{y}$, that the greatest positive value of y will correspond to the smallest of x; hence, designating the superior limit of the positive roots of the equation $Y = 0$ by L, we shall have $\frac{1}{L} = L'$, *the inferior limit of the positive roots of the given equation.*

2d. If in the equation $X = 0$, we make $x = -y$, which gives the transformed equation $Y = 0$, it is clear that the positive roots of this new equation, taken with the sign $-$, will give the negative roots of the given equation; therefore, determining, by the known methods, the superior limit L of the positive roots of the equation $Y = 0$, we shall have $-L = L''$, the *superior limit* (numerically) *of the negative roots of the proposed equation.*

3d. Finally, if we replace x, in the given equation, by $-\frac{1}{y}$, and find the superior limit L of the transformed equation $Y=0$, then, $L''' = -\frac{1}{L}$ will be *the inferior limit* (numerically) *of the negative roots of the given equation.*

Consequences deduced from the preceding Principles.

First.

320. *Every equation in which there are no variations in the signs,* that is, *in which all the terms are positive, must have all of its real roots negative;* for, every positive number substituted for x, will render the first member essentially positive.

Second.

321. *Every complete equation,* having its terms alternately positive and negative, must have its real roots all positive; for, every negative number substituted for x in the proposed equation, would render all the terms positive, if the equation was of an even degree, and all of them negative, if it were of an odd degree. Hence, their sum could not be equal to zero in either case.

This principle is also true for *every incomplete equation, in which there results, by substituting* $-y$ *for* x, *an equation having all of its terms affected with the same sign.*

Third.

322. Every equation of an odd degree, the co-efficients of which are real, *has at least one real root affected with a sign contrary to that of its last term.*

For, let

$$x^m + Px^{m-1} + \dots Tx \pm U = 0,$$

be the proposed equation; and first consider the case in which the last term is *negative.*

By making $x=0$, the first member becomes $-U$. But by giving a value to x equal to the greatest co-efficient plus unity, or $(K+1)$, the first term x^m will become greater than the arithmetical sum of all the others (Art. 315), the result of this substitution will therefore be *positive;* hence, *there is at least one*

real root comprehended between 0 *and* $K+1$, which root is positive, and consequently affected with a sign *contrary* to that of the last term.

Suppose now, that the last term is *positive*.

Making $x=0$, we obtain $+U$ for the result; but by putting $-(K+1)$ in place of x, we shall obtain a *negative* result, since the first term becomes negative by this substitution; hence, the equation has at least one real root comprehended between 0 and $-(K+1)$, which is negative, or *affected with a sign contrary* to that of the last term.

Fourth.

323. *Every equation of an even degree,* involving only real coefficients *of which the last term is negative, has at least two real roots, one positive and the other negative.*

For, let $-U$ be the last term; making $x=0$, there results $-U$. Now substitute either $K+1$, or $-(K+1)$, K being the greatest co-efficient in the equation. As m is an even number, the first term x^m will remain positive; besides, by these substitutions, it becomes greater than the sum of all the others; therefore, the results obtained by these substitutions are both *positive*, or affected with a sign contrary to that given by the hypothesis $x=0$; hence, the equation *has at least two real roots*, one *positive*, and comprehended between 0 and $K+1$, the other *negative*, and comprehended between 0 and $-(K+1)$.

Fifth.

324. *If an equation, involving only real co-efficients, contains imaginary roots, the number of such roots must be even.*

For, conceive that the first member has been divided by all the simple factors corresponding to the real roots; the co-efficients of the *quotient* will be real (Art. 278); *and the equation must also be of an even degree;* for, if it was uneven, by placing it equal to zero, we should obtain an equation that would contain at least one real root; hence, the imaginary roots must enter by pairs.

REMARK.—325. There is a property of the above polynomial quotient which belongs exclusively to equations containing only imaginary roots; viz., *every such equation always remains positive for any real value substituted for* **x**.

For, by substituting for x, $K+1$, the greatest co-efficient plus unity, we could always obtain a positive result; hence, if the polynomial could become negative, it would follow that when placed equal to zero, there would be at least one real root comprehended between $K+1$ and the number which would give a negative result (Art. 311).

It also follows, that the last term of this polynomial must be *positive*, otherwise $x=0$ would give a negative result.

Sixth.

326. *When the last term of an equation is positive, the number of its real positive roots is even; and when it is negative, the number of such roots is uneven.*

For, first suppose that the last term is $+U$, or *positive*. Since by making $x=0$, there will result $+U$, and by making $x=K+1$, the result will also be positive, it follows that 0 and $K+1$ give two results affected with the same sign, and consequently (Art. 313), the number of real roots, if any, comprehended between them, is even.

When the last term is $-U$, then 0 and $K+1$, give two results affected with contrary signs, and consequently comprehend either a *single root*, or *an odd number of them.*

The *reciprocal* of this proposition is evident.

Descartes' Rule.

327. *An equation of any degree whatever, cannot have a greater number of positive roots than there are* variations *in the signs of its terms, nor a greater number of negative roots than there are* permanences *of these signs.*

A *variation* is a change of sign in passing along the terms, and a *permanence* is when two consecutive terms have the same sign.

In the equation $x-a=0$, there is one variation, and one positive root, $x=a$. And in the equation $x+b=0$, there is one permanence, and one negative root, $x=-b$.

If these equations be multiplied together, there will result an equation of the second degree,

$$\left.\begin{array}{r|l} x^2 - a & x - ab \\ + b & \end{array}\right\} = 0.$$

If a is less than b, the equation will be of the first form (Art. 144); and if $a > b$ the equation will be of the second form; that is,

$$a < b \quad \text{gives} \quad x^2 + 2px - q = 0,$$

and

$$a > b \quad \text{"} \quad x^2 - 2px - q = 0.$$

In the first case, there is one permanence, and one variation, and in the second, one variation and one permanence. Since in either form, one root is positive and one negative, it follows that there are as many positive roots as there are variations, and as many negative roots as there are permanences.

The proposition will evidently be demonstrated in a general manner, if it be shown that the multiplication of the first member by a factor $x - a$, corresponding to a *positive* root, introduces *at least one variation*, and that the multiplication by a factor $x + a$, corresponding to a negative root, introduces *at least one permanence.*

Take the equation

$$x^m \pm Ax^{m-1} \pm Bx^{m-2} \pm Cx^{m-3} \pm \ldots \pm Tx \pm U = 0,$$

in which the signs succeed each other in any manner whatever. By multiplying by $x - a$, we have

$$\left.\begin{array}{l|l|l|l|l} x^{m+1} \pm A & x^m \pm B & x^{m-1} \pm C & x^{m-2} \pm \ldots \pm U & x \qquad \\ \quad\;\; - a & \quad \mp Aa & \quad\;\; \mp Ba & \qquad\qquad\;\; \mp Ta & \;\; \mp Ua \end{array}\right\} = 0.$$

The co-efficients which form the first horizontal line of this product, are those of the given equation, taken with the same sign; and the co-efficients of the second line are formed from those of the first, by multiplying by a, changing the signs, and advancing each one place to the right.

Now, so long as each co-efficient of the upper line is greater than the corresponding one in the lower, it will determine the sign of the total co-efficient; hence, in this case there will be, from the first term to that preceding the last, inclusively, the same variations and the same permanences as in the proposed equation; but the last term, $\mp Ua$ having a sign contrary to that which immediately precedes it, there must be one more variation than in the proposed equation.

When a co-efficient in the lower line is affected with a sign contrary to the one corresponding to it in the upper, and is also greater than this last, there is a change from a permanence of sign to a variation; for the sign of the term in which this happens,

being the same as that of the inferior co-efficient, must be contrary to that of the preceding term, which has been supposed to be the same as that of its superior co-efficient. Hence, each time we descend from the upper to the lower line, in order to determine the sign, there is a variation which is not found in the proposed equation; and if, after passing into the lower line, we continue in it throughout, we shall find for the remaining terms the same variations and the same permanences as in the given equation, since the co-efficients of this line are all affected with signs contrary to those of the primitive co-efficients. This supposition would therefore give us one variation for each positive root. But if we ascend from the lower to the upper line, there may be either a variation or a permanence. But even by supposing that this passage produces permanences in all cases, since the last term $\mp Ua$ forms a part of the lower line, it will be necessary to go once more from the upper line to the lower, than from the lower to the upper. Hence, the new equation *must have at least one more variation than the proposed;* and it will be the same for each positive root introduced into it.

It may be demonstrated, in an analogous manner, that *the multiplication by a factor $x + a$,* corresponding to a negative root, *would introduce one permanence more.* Hence, in any equation, the number of positive roots cannot be greater than the number of VARIATIONS of signs, nor the number of negative roots greater than the number of PERMANENCES.

Consequence.

328. When the roots of an equation are all real, *the number of positive roots is equal to the number of variations, and the number of negative roots to the number of permanences.*

For, let m denote the degree of the equation, n the number of variations of the signs, p the number of permanences; we shall have $m = n + p$. Moreover, let n' denote the number of positive roots, and p' the number of negative roots, we shall have

$$m = n' + p';$$

whence $\quad n + p = n' + p', \quad$ or, $\quad n - n' = p' - p.$

Now, we have just seen that n' cannot be $> n$, nor can it be less, since p' cannot be $> p$; therefore we must have $n' = n$, and $p' = p$.

REMARK.—329. When an equation wants some of its terms, we can often discover the presence of imaginary roots, by means of the above rule.

For example, take the equation

$$x^3 + px + q = 0,$$

p and q being essentially positive; introducing the term which is wanting, by affecting it with the co-efficient ± 0: it becomes

$$x^3 \pm 0 \,.\, x^2 + px + q = 0.$$

By considering only the superior sign, we should only obtain permanences, whereas the inferior sign gives two variations. This proves that the equation has some imaginary roots; for, if they were all three real, it would be necessary by virtue of the superior sign, that they should be all negative, and, by virtue of the inferior sign, that two of them should be positive and one negative, which are *contradictory results*.

We can conclude nothing from an equation of the form

$$x^3 - px + q = 0;$$

for, introducing the term $\pm 0 \,.\, x^2$, it becomes

$$x^3 \pm 0 \,.\, x^2 - px + q = 0,$$

which contains one permanence and two variations, whether we take the superior or inferior sign. Therefore, this equation may have its three roots real, viz., two positive and one negative; or, two of its roots may be imaginary and one negative, since its last term is positive (Art. 326).

Of the commensurable Roots of Numerical Equations.

330. Every equation in which the co-efficients are whole numbers, that of the first term being unity, will have whole numbers only for its commensurable roots.

For, let there be the equation

$$x^m + Px^{m-1} + Qx^{m-2} + \dots + Tx + U = 0;$$

in which $P, Q \dots T, U$, are whole numbers, and suppose that it were possible for one root to be a commensurable fraction $\frac{a}{b}$. Substituting this fraction for x, the equation becomes

$$\frac{a^m}{b^m} + P\frac{a^{m-1}}{b^{m-1}} + Q\frac{a^{m-2}}{b^{m-2}} + \dots + T\frac{a}{b} + U = 0;$$

whence, multiplying both members by b^{m-1}, and transposing,

$$\frac{a^m}{b} = -Pa^{m-1} - Qa^{m-2}b - \ldots - Tab^{m-2} - Ub^{m-1}.$$

But the second member of this equation is composed of a series of entire numbers, while the first is essentially fractional, for a and b being prime with each other, a^m and b will also be prime with each other (Art. 118), and hence this equality cannot exist; for, an irreducible fraction cannot be equal to a whole number.

Therefore, it is impossible for any commensurable fraction to satisfy the equation. Now, it has been shown (Art. 294), that an equation containing rational, but fractional co-efficients, can be transformed into another in which the co-efficients are whole numbers, that of the first term being unity. Hence *the research of the commensurable roots*, either entire or fractional, *can always be reduced to that of the entire roots.*

331. This being the case, take the general equation

$$x^m + Px^{m-1} + Qx^{m-2} \ldots + Rx^3 + Sx^2 + Tx + U = 0,$$

and let a denote any entire number, positive or negative, which will verify it.

Since a is a root, we shall have the equation

$$a^m + Pa^{m-1} + \ldots + Ra^3 + Sa^2 + Ta + U = 0 \ldots (1).$$

Replace a by all the entire positive and negative numbers between 1 and the limit $+L$, and between -1 and $-L''$: those which verify the above equality will be roots of the equation. But these trials being long and troublesome, we will deduce from equation (1), other conditions equivalent to this, and easier verified.

Transposing all the terms except the last, and dividing by a, equation (1) becomes

$$\frac{U}{a} = -a^{m-1} - Pa^{m-2} - \ldots - Ra^2 - Sa - T \ldots (2).$$

Now, the second member of this equation is an entire number; hence $\frac{U}{a}$ must be an entire number; therefore, *the entire roots of the equation are comprised among the divisors of the last term.*

Transposing $-T$ in equation (2), dividing by a, and making

$$\frac{U}{a} + T = T', \text{ we have,}$$

$$\frac{T'}{a} = -a^{m-2} - Pa^{m-3} \ldots - Ra - S \ldots (3).$$

The second member of this equation being entire, $\frac{T'}{a}$, that is, *the quotient of the division of*

$$\frac{U}{a} + T \text{ by } a,$$

is an entire number.

Transposing the term $-S$ and dividing by a, we have, by supposing

$$\frac{T'}{a} + S = S'.$$

$$\frac{S'}{a} = -a^{m-3} - Pa^{m-4} - \ldots - R \ldots (4).$$

The second member of this equation being entire, $\frac{S'}{a}$, that is, *the quotient of the division of*

$$\frac{T'}{a} + S \text{ by } a,$$

is an entire number.

By continuing to transpose the terms of the second member into the first, we shall, after $m-1$ transformations, obtain an equation of the form

$$\frac{Q'}{a} = -a - P.$$

Then, transposing the term $-P$, dividing by a, and making

$$\frac{Q'}{a} + P = P', \text{ we have } \frac{P'}{a} = -1, \text{ or } \frac{P'}{a} + 1 = 0.$$

This equation, which is only a transformation of equation (1), is the *last condition which it is requisite for* the entire number a to satisfy, in order that it may be known to be a root of the equation.

332. From the preceding conditions we conclude that, when an entire number a, positive or negative, is a root of the given equation, *the quotient of the last term, divided by a, is an entire number.*

Adding to this quotient the co-efficient of x^1, *the quotient of this sum, divided by a, must also be entire.*

Adding the co-efficient of x^2 to this last quotient, and again dividing by a, *the new quotient must also be entire;* and so on.

Finally, adding the co-efficient of the second term, that is, of x^{m-1}, to the preceding quotient, *the quotient of this sum divided by* a, *must be equal to* -1; hence, *the result of the addition of unity, which is the co-efficient of* x^m, *to the preceding quotient, must be equal to* 0.

Every number which will satisfy these conditions will be a root, and those which do not satisfy them should be rejected.

All the entire roots may be determined at the same time, as follows:

After having determined all the divisors of the last term, write those which are comprehended between the limits $+L$ *and* $-L''$ *upon the same horizontal line; then underneath these divisors write the quotients of the last term by each of them.*

Add the co-efficient of x^1 *to each of these quotients, and write the sums underneath the quotients which correspond to them. Then divide these sums by each of the divisors, and write the quotients underneath the corresponding sums, taking care to reject the fractional quotients and the divisors which produce them;* and so on.

When there are terms wanting in the proposed equation, their co-efficients, which are to be regarded as equal to 0, must be taken into consideration.

EXAMPLES.

1. What are the entire roots of the equation

$$x^4 - x^3 - 13x^2 + 16x - 48 = 0\,?$$

The superior limit of the positive roots of this equation (Art 317), is $13 + 1 = 14$. The co-efficient 48 need not be considered, since the last two terms can be put under the form $16(x - 3)$; hence, when $x > 3$, this part is essentially positive.

The superior limit of the negative roots (Art. 319), is

$$-(1 + \sqrt{48}), \quad \text{or} \quad -8.$$

Therefore, the divisors of the last term which may give roots, are 1, 2, 3, 4, 6, 8, 12; moreover, neither $+1$, nor -1, will satisfy the equation, because the co-efficient -48 is itself greater than the sum of all the others: we should therefore try only the *positive divisors* from 2 to 12, and the *negative divisors* from -2 to -6 inclusively.

By observing the rule given above, we have

12,	8,	6,	4,	3,	2,	$-$ 2,	$-$ 3,	$-$ 4,	$-$ 6
$-$ 4,	$-$ 6,	$-$ 8,	$-$ 12,	$-$ 16,	$-$ 24,	$+$ 24,	$+$ 16,	$+$ 12,	$+$ 8
$+$ 12,	$+$ 10,	$+$ 8,	$+$ 4,	0,	$-$ 8,	$+$ 40,	$+$ 32,	$+$ 28,	$+$ 24
$+$ 1,	..,	..,	$+$ 1,	0,	$-$ 4,	$-$ 20,	..,	$-$ 7,	$-$ 4
$-$ 12,	..,	..,	$-$ 12,	$-$ 13,	$-$ 17,	$-$ 33,	..,	$-$ 20,	$-$ 17
$-$ 1,	..,	..,	$-$ 3,	..,	..,	..,	..,	$+$ 5,	..
$-$ 2,	..,	..,	$-$ 4,	..,	..,	..,	..,	$+$ 4,	..
..,	..,	..,	$-$ 1,	..,	..,	..,	..,	$-$ 1,	..

The *first* line contains the divisors, the *second* contains the quotients arising from the division of the last term -48, by each of the divisors. The *third* line contains these quotients augmented by the co-efficient $+16$, and the *fourth*, the quotients of these sums by each of the divisors; this second condition excludes the divisors $+8$, $+6$, and -3.

The *fifth* is the preceding line of quotients, augmented by the co-efficient -13, and the *sixth* is the quotients of these sums by each of the divisors; this third condition excludes the divisors 3, 2, -2, and -6.

Finally, the *seventh* is the third line of quotients, augmented by the co-efficient -1, and the *eighth* is the quotients of these sums by each of the divisors. The divisors $+4$ and -4 are the only ones which give -1; hence, $+4$ and -4 are the only entire roots of the equation.

In fact, if we divide

$$x^4 - x^3 - 13x^2 + 16x - 48,$$

by the product $(x - 4)(x + 4)$, or $x^2 - 16$, the quotient will be $x^2 - x + 3$, which placed equal to zero, gives

$$x = \frac{1}{2} \pm \frac{1}{2}\sqrt{-11};$$

therefore, the four roots are

$$4, \quad -4, \quad \frac{1}{2} + \frac{1}{2}\sqrt{-11} \quad \text{and} \quad \frac{1}{2} - \frac{1}{2}\sqrt{-11}.$$

2. What are the entire roots of the equation

$$x^4 - 5x^3 + 25x - 21 = 0?$$

3. What are the entire roots of the equation

$$15x^5 - 19x^4 + 6x^3 + 15x^2 - 19x + 6 = 0\,?$$

4. What are the entire roots of the equation

$$9x^6 + 30x^5 + 22x^4 + 10x^3 + 17x^2 - 20x + 4 = 0.$$

Sturms' Theorem.

333. The object of this theorem is to explain a method of determining the number and places of the real roots of equations involving but one unknown quantity. Let

$$X = 0 \;.\;.\;.\;.\; (1),$$

represent an equation containing the single unknown quantity x; X being a polynomial of the m^{th} degree with respect to x, the co-efficients of which are all real. If this equation should have equal roots, they may be found and divided out as in Art. 302, and the following reasoning be applied to the equation which would result. We will therefore suppose $X = 0$ to have no equal roots.

334. Let us denote the first-derived polynomial of X by X_1, and then apply to X and X_1 a process similar to that for finding their greatest common divisor, differing only in this respect, that instead of using the successive remainders as at first obtained, *we change their signs, and take care also, in preparing for the division, neither to introduce nor reject any factor except a positive one.*

If we denote the several remainders, in order, after their signs have been changed, by $X_2, X_3 \ldots X_r$, which are read X second, X third, &c., and denote the corresponding quotients by Q_1, $Q_2 \ldots Q_{r-1}$, we may then form the equations

$$X = X_1 Q_1 - X_2 \;.\;.\;.\;.\; (2),$$

$$\left.\begin{array}{l} X_1 = X_2 Q_2 - X_3 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ X_{n-1} = X_n Q_n - X_{n+1} \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ X_{r-2} = X_{r-1} Q_{r-1} - X_r \end{array}\right\} \;.\;.\;.\; (3).$$

Since by hypothesis, $X = 0$ has no equal roots, no common divisor can exist between X and X_1 (Art. 300). The last remainder $-X_r$, will therefore *be different from zero,* and *independent* of x.

335. Now, let us suppose that a number p has been substituted for x in each of the expressions X, X_1, $X_2 \ldots X_{r-1}$; and that the signs of the results, together with the sign of X_r, are arranged in a line one after the other: also that another number q, greater than p, has been substituted for x, and the signs of the results arranged in like manner.

Then will the number of variations in the signs of the first arrangement, diminished by the number of variations in those of the second, denote the exact number of real roots comprised between p *and* q.

336. The demonstration of this truth mainly depends upon the four following properties of the expressions X, $X_1 \ldots X_n$, &c.

I. Let a be a root of the equation $X = 0$. If we substitute $a + u$ for x, and designate by A what X becomes, and denote the derived polynomials by A', A'', A''', &c.; we shall have (Art. 299),

$$A + A'u + \frac{A''}{2}u^2 \ldots\ldots + u^m.$$

But since by hypothesis, a is a root of the equation $X = 0$, we have $A = 0$, and hence the above expression becomes

$$u\left(A' + \frac{A''}{2}u + \frac{A'''}{2.3}u^2 \ldots\ldots + u^{n-1}\right);$$

in which A' is not zero, since the equation $X = 0$ is supposed not to contain equal roots. Now we say, *that* u *can be made so small, that the sign of the quantity within the parenthesis shall be the same as that of its first term.*

We attain this object, by finding for u a value which shall render, numerically,

$$A' > \frac{A''}{2}u + \frac{A'''}{2.3}u^2 + \&c. \ldots\ldots u^{n-1};$$

that is, $$A' > u\left(\frac{A''}{2} + \frac{A'''}{2.3}u + \&c. \ldots\ldots u^{n-2}\right);$$

a condition which will always be fulfilled (Art. 315), when

$$u = \text{ or } < \frac{A'}{K+A'}, \; K \text{ being the greatest co-efficient of } u.$$

II. *If any number be substituted for* x *in these expressions, it is impossible that any two consecutive ones can become zero at the same time.*

For, let X_{n-1}, X_n, X_{n+1}, be any three consecutive expressions. Then among equations (3), we shall find

$$X_{n-1} = X_n Q_n - X_{n+1} \;\ldots\; (4),$$

from which it appears that, if X_{n-1} and X_n should both become 0 for a value of x, X_{n+1} would be 0 for the same value; and since the equation which follows (4) must be

$$X_n = X_{n+1} Q_{n+1} - X_{n+2},$$

we shall have $X_{n+2} = 0$ for the same value, and so on until we should find $X_r = 0$, which cannot be; hence, X_{n-1} and X_n cannot both become 0 for the same value of x.

III. By an examination of equation (4), we see that if X_n becomes 0 for a value of x, X_{n-1} and X_{n+1} must have contrary signs; that is, *if any one of the expressions is reduced to* 0 *by the substitution of a value for* x, *the preceding and following ones will have contrary signs for the same value.*

IV. Let us substitute $a + u$ for x in the expressions X and X_1, and designate by U and U_1 what they respectively become under this supposition. Then (Art. 297), we have

$$\left.\begin{aligned} U &= A + A'u + A''\frac{u^2}{2} + \&c. \\ U_1 &= A_1 + A'_1 u + A''_1\frac{u^2}{2} + \&c. \end{aligned}\right\} \;\ldots\; (5),$$

in which A, A', A'', &c., are the results obtained by the substitution of a for x, in X and its derived polynomials; and A_1 A'_1 &c., are similar results derived from X_1. If now, a be a root of the proposed equation $X = 0$, then $A = 0$, and since A' and A_1 are each derived from X_1, by the substitution of a for x, we have $A' = A_1$, and equations (5) become

$$\left.\begin{aligned} U &= A'u + A''\frac{u^2}{2} + \&c. \\ U_1 &= A' + A'_1 u \quad + \&c. \end{aligned}\right\} \;\ldots\; (6).$$

Now, the arbitrary quantity u may be taken so small that when added to a, it will but insensibly increase it, and when subtracted from a, it will but insensibly diminish it; in which cases, the signs of the values of U and U_1 will depend upon the signs of their first terms; that is, they will be alike when u is positive or when $a + u$ is substituted for x, and unlike when u is negative or when

$a - u$ is substituted for x. Hence, *if a number insensibly less than one of the real roots of $X = 0$ be substituted for* x *in X and X_1, the results will have contrary signs, and if a number insensibly greater than this root be substituted, the results will have the same sign.*

337. Now, let any number as k, algebraically less, that is, nearer equal to $-\infty$, than any of the real roots of the several equations

$$X = 0, \quad X_1 = 0 \ldots X_{r-1} = 0,$$

be substituted for x in them, and the signs of the several results arranged in order; then, let x be increased by insensible degrees, until it becomes equal to h the least of all the roots of the equations. As there is no root of either of the equations between k and h, none of the signs can change while x is less than h (Art. 311), and the number of variations and permanences in the several sets of results, will remain the same as in those obtained by the first substitution.

When x becomes equal to h, *one* or *more* of the expressions X, X_1, &c., will reduce to 0. Suppose X_n becomes 0. Then, as by the second and third properties above explained, neither X_{n-1} nor X_{n+1} can become 0 at the same time, but *must have* contrary signs, it follows that in passing from one to the other (omitting $X_n = 0$), there will be *one* and *only one* variation and since their signs have not changed, one must be *the same as*, and the other *contrary to*, that of X_n, both before and after it becomes 0; hence, in passing over the three, either just before X_n becomes 0 or just after, there is *one* and *only one* variation. Therefore, the reduction of X_n to 0 neither increases nor diminishes the number of variations; and this will evidently be the case, although several of the expressions X_1, X_2, &c., should become 0 at the same time.

If $x = h$ should reduce X to 0, then h is the least real root of the proposed equation, which root we denote by a; and since by the fourth property, just before x becomes equal to a, the signs of X and X_1 are contrary, *giving a variation*, and just after passing it (before x becomes equal to a root of $X_1 = 0$), the signs are the same, *giving a permanence instead*, it follows that in passing this root *a variation is lost*. In the same way, increasing x by insensible degrees from $x = a + u$ until we reach the root of $X = 0$ next in order, it is plain that no variation will be lost or gained in passing any of the roots of the other equations, but that

in passing this root, for the same reason as before, another variation will be lost, and so on for each real root between k and the number last substituted, as g, a variation will be lost until x has been increased beyond the greatest real root, *when no more can be lost or gained*. Hence, the *excess* of the number of variations obtained by the substitution of k over those obtained by the substitution of g, will be equal to the number of real roots comprised between k and g.

It is evident that the same course of reasoning will apply when we commence with any number p, whether less than all the roots or not, and gradually increase x until it equals any other number q. The fact enunciated in Art. 335 is therefore established.

338. In seeking the number of roots comprised *between* p and q, should either p or q reduce any of the expressions X_1, X_2, &c., to 0, the result will not be affected by their omission, since the number of variations will be the same.

Should p reduce X to 0, then p is a root, but not one of those sought; and as the substitution of $p+u$ will give X and X_1 the same sign, the number of variations to be counted will not be affected by the omission of $X=0$.

Should q reduce X to 0, then q is also a root; and as the substitution of $q-u$ will give X and X_1 contrary signs, one variation must be counted in passing from X to X_1.

339. If in the application of the preceding principles, we observe that any one of the expressions X_1, X_2 . . . &c., X_n for instance, will preserve the same sign for all values of x in passing from p to q, inclusive, it will be unnecessary to use the succeeding expressions, or even to deduce them. For, as X_n preserves the same sign during the successive substitutions, it is plain that the same number of variations will be lost among the expressions X, X_1, &c. . . . ending with X_n as among all including X_r. Whenever then, in the course of the division, it is found that by placing any of the remainders equal to 0, an equation is obtained with imaginary roots only (Art. 325), it will be useless to obtain any of the succeeding remainders. This principle will be found very useful in the solution of numerical examples.

340. As all the real roots of the proposed equation are necessarily included between $-\infty$ and $+\infty$, we may, by ascertaining

the number of variations lost by the substitution of these, in succession, in the expressions X, $X_1 \ldots X_n, \ldots$ &c., readily determine the total number of such roots. It should be observed, that it will be only necessary to make these substitutions in the first terms of each of the expressions, as in this case the sign of the term will determine that of the entire expression (Art. 315).

341. Having thus obtained the total number of real roots, we may ascertain their places by substituting for x, in succession, the values 0, 1, 2, 3, &c., until we find an entire number which gives the same number of variations as $+\infty$. This will be the smallest superior limit of the positive roots in entire numbers.

Then substitute 0, -1, -2, &c., until a negative number is obtained which gives the same number of variations as $-\infty$ This will be, numerically, the smallest superior limit of the negative roots in entire numbers. Now, by commencing with this limit and observing the number of variations lost in passing from each number to the next in order, we shall discover how many roots are included between each two of the consecutive numbers used, and thus, of course, know the entire part of each root. The decimal part may then be sought by some of the known methods of approximation.

EXAMPLES.

1. Let $$8x^3 - 6x - 1 = 0 = X.$$

The first-derived polynomial (Art. 297), is

$$24x^2 - 6,$$

and since we may omit the positive factor 6, without affecting the sign, we may write

$$4x^2 - 1 = X_1.$$

Dividing X by X_1, we obtain for the first remainder, $-4x - 1$. Changing its sign, we have

$$4x + 1 = X_2.$$

Multiplying X_1 by the positive number 4, and then dividing by X_2, we obtain the second remainder -3; and by changing its sign

$$+3 = X_3.$$

The expressions to be used are then

$$X = 8x^3 - 6x - 1, \quad X_1 = 4x^2 - 1, \quad X_2 = 4x + 1, \quad X_3 = +3.$$

Substituting $-\infty$ and then $+\infty$, we obtain the two following arrangements of signs:

$$- + - + \ldots\ldots 3 \textit{ variations},$$

$$+ + + + \ldots\ldots 0 \quad \text{"}$$

There are then *three* real roots.

If now, in the same expressions we substitute 0 and $+1$, and then 0 and -1, for x, we shall obtain the three following arrangements:

For	$x = +1$	$+ + + +$	0 *variations*,
"	$x = 0$	$- - + +$	1 "
"	$x = -1$	$- + - +$	3 "

As $x = +1$ gives the same number of variations as $+\infty$, and $x = -1$ gives the same as $-\infty$, $+1$ and -1 are the smallest limits in entire numbers. In passing from -1 to 0, *two* variations are lost, and in passing from 0 to $+1$, *one* variation is lost; hence, there are two negative roots between -1 and 0, and one positive root between 0 and $+1$.

2. Let $\qquad 2x^4 - 13x^2 + 10x - 19 = 0.$

If we deduce X, X_1, and X_2, we have the three expressions

$$X = 2x^4 - 13x^2 + 10x - 19,$$

$$X_1 = 4x^3 - 13x + 5,$$

$$X_2 = 13x^2 - 15x + 38.$$

If we place $X_2 = 0$, we shall find that both of the roots of the resulting equation are imaginary; hence, X_2 will be positive for all values of x (Art. 325). It is then useless to seek for X_3 and X_4.

By the substitution of $-\infty$ and $+\infty$ in X, X_1, and X_2, we obtain for the first, *two* variations, and for the second *none*; hence, there are two real and two imaginary roots in the proposed equation.

3. Let $\qquad x^3 - 5x^2 + 8x - 1 = 0.$

4. $\qquad x^4 - x^3 - 3x^2 + x^2 - x - 3 = 0.$

5. $\qquad x^5 - 2x^3 + 1 = 0.$

Discuss each of the above equations.

Young's Method of resolving Cubic Equations.

342. Every numerical equation of the third degree may be reduced to the form

$$x^3 + bx^2 + cx = N \qquad (1),$$

in which b, c, and N, are known numbers.

Since this equation will have at least one real root (Art. 277), let us find, either by Sturms' theorem or by trial, two consecutive numbers, either integral or decimal, which being substituted for x, will give results with different signs. We then know that one of the values of x will lie between them (Art. 311), and consequently, that the smallest number will be the first figure of the required root.

Let us designate this figure by r. Now, if we neglect the remaining figures of the root, and regard r as the approximate value of x, we shall have

$$r^3 + br^2 + cr = N; \text{ whence, } \quad r = \frac{N}{r^2 + br + c}.$$

Having found r, denote the remaining part of the root by y; then

$$x = r + y.$$

Substituting this value of x in the given equation, we have

$$\left.\begin{aligned} x^3 &= r^3 + 3r^2y + 3ry^2 + y^3 \\ bx^2 &= br^2 + 2bry + by^2 \\ cx &= cr + cy \end{aligned}\right\} = N$$

and by adding and arranging with reference to y,

$$y^3 + (3r + b)\,y^2 + (3r^2 + 2br + c)\,y + (r^3 + br^2 + cr) = N.$$

But we may simplify the form of this equation, by making

$$b' = 3r + b, \quad c' = 3r^2 + 2br + c, \quad N' = N - (r^3 + br^2 + cr);$$

which will give $\quad y^3 + b'y^2 + c'y = N' \qquad (2).$

The form of this equation is entirely similar to that of the given equation; and if we denote by s the first figure of y and make the same supposition as before, we shall have

$$s^3 + b's^2 + c's = N';$$

whence,
$$s = \frac{N'}{s^2 + b's + c'}.$$

Supposing the value of s found; denote by z the remaining figures of the root: then

$$y = s + z,$$

and

$$y^3 = s^3 + 3s^2z + 3sz^2 + z^3,$$

$$b'y^2 = b's^2 + 2b'sz + b'z^2,$$

$$c'y = c's + c'z.$$

By adding and arranging with reference to z,

$$z^3 + (3s + b')z^2 + (3s^2 + 2b's + c')z + (s^3 + b's^2 + c's) = N'.$$

But, by making $\quad b'' = 3s + b',$

$$c'' = 3s^2 + 2b's + c',$$

$$N'' = N' - (s^3 + b's^2 + c's),$$

the equation becomes

$$z^3 + b''z^2 + c''z = N'' \qquad (3).$$

If we denote by t the first figure of z, and make the same supposition as before, we shall have

$$t^3 + b''t^2 + c''t = N'';$$

whence,

$$t = \frac{N''}{t^2 + b''t + c''}.$$

If we designate by u the first figure of t, we should find, by a process similar to the above,

$$u = \frac{N'''}{u^2 + b'''u + c'''};$$

and in a similar manner we may find the algebraic expression for any succeeding figure of the root.

343. It is now required to put these algebraic expressions under such a form as will indicate the best practical rule for performing the arithmetical operations. For this purpose, let us bring the formulas together. We have

$$r = \frac{N}{r^2 + br + c} = \frac{N}{r(r + b) + c};$$

$$s = \frac{N'}{s^2 + b's + c'} = \frac{N'}{s(s + 3r + b) + 3r^2 + 2br + c};$$

$$t = \frac{N''}{t^2 + b''t + c''} = \frac{N''}{t[t + 3(r + s) + b] + 3s^2 + 2b's + c'};$$

$$u = \frac{N'''}{u^2 + b'''u + c'''} = \frac{N'''}{u[u + 3(r + s + t) + b] + 3t^2 + 2b''t + c''}.$$

&c., &c., &c.

The value of r being found, and c a known number, the denominator $r(r+b)+c$ will be known. This forms the first divisor, and dividing N by it, the first figure of the quotient will be r, as before found. Multiplying the divisor by r and subtracting the product from N, we obtain N', the second dividend.

It will be seen that the three right-hand terms in each denominator, are formed from the preceding figure of the root. These make *trial divisors* for each figure of the root after the first.

Having found N', we form its trial divisor and then see how often it is contained in N', which gives s. We then form the complete divisor which we multiply by s, and subtract the result from N', which gives N''. We then form its trial divisor, find the figure t, after which we find the complete divisor for t, and then multiply it by the quotient figure t and subtract the result from N'', giving N'''; and similarly for all the following figures of the root.

344. By examining the table of Example I, on the next page, and comparing it with the formulas, we see, that if under any divisor we write the square of the figure of the root which the divisor determines, and then add it to the two numbers directly above, their sum will be the *trial divisor* for the next figure of the root. Hence, we have the following

RULE.

I. *Write down* c, *the co-efficient of* x, *and on the same line, to the right, place the known number N, and set in the quotient the first figure of the root found by trial.*

II. *To this figure of the root add* b, *the co-efficient of* x^2, *and then multiply the sum by the figure of the root, and add the product to* c, *and the sum will be the first divisor, which is then to be multiplied by the quotient figure and the product subtracted from N.*

III. *Under the first divisor write the square of the first figure of the root, and then add it to the last two sums, and the result will be the trial divisor for the next figure of the root.*

IV. *Having found the next figure of the root, add to it three times the figures of the root already found, and also the co-efficient* b: *then multiply the sum by the figure of the root and add the product to the trial divisor, and the sum will be the entire divisor, which must then be multiplied by the figure of the root, and the product subtracted from the last dividend. The process for determining other figures of the root is entirely similar.*

c	$=$	6
$r(r+b) = 2(2+8)$	$=$	20
1st divisor $= r(r+b) + c$	$=$	26
r^2	$=$	4
1st trial divisor $= 3r^2 + 2br + c = c'$	$=$	50
$s(s + 3r + b)$	$=$	5.76
2d divisor $= s(s + 3r + b) + c'$	$=$	55.76
s^2	$=$	$.16$
2d trial divisor $= 3s^2 + 2b's + c' = c''$	$=$	61.68
$t[t + 3(r+s) + b]$	$=$	$.3044$
3d divisor $= t[t + 3(r+s) + b] + c''$	$=$	61.9844
t^2	$=$	0.0004
3d trial divisor $= 3t^2 + 2b''t + c'' = c'''$	$=$	62.2892
$u[u + 3(r+s+t) + b]$	$=$	0.076325
4th divisor $= u[u+3(r+s+t)+b] + c'''$	$=$	$62.365525.$

&c.

Hence, one of the roots of the given equation is

$$2.425+.$$

$$N = 75.9 \,|\, \underset{}{\overset{r}{2}}.\overset{s}{4}\overset{t}{2}\overset{u}{5}+ = x, \text{ or one of the roots.}$$

52	$= r^3 + br^2 + cr =$ 1st subtrahend.
23.9	$= N' =$ 2d dividend.
22.304	$= s^3 + b's^2 + c's =$ 2d subtrahend.
1.596	$= N'' =$ 3d dividend.
1.239688	$= t^3 + b''t^2 + c''t =$ 3d subtrahend.
0.356312	$= N''' =$ 4th dividend.
0.311827625	$= u^3 + b'''u^2 + c'''u =$ 4th subtrahend.
0.044484375	$= N'''' =$ 5th dividend.

&c.

EXAMPLE.

1. Find one value of x in the equation

$$x^3 + \underset{b}{8}x^2 + \underset{c}{6}x = \underset{N}{75.9}.$$

REMARK.—The operations in the example of the table, are all performed according to the directions of the rule; but more decimal places have been used in the dividends and divisors, in the latter part of the work, than were necessary. Had we admitted but three places of decimals in the dividends, and rejected all other places to the right, as fast as they occurred, we should still have had the root equally true to at least four places of decimals. But since the figures of the root are decimals, it follows that if the number of decimal places in the dividend does not exceed three, the decimal places in the corresponding divisor should not exceed two; and for every succeeding figure found in the root, one place may be struck off from the right of the divisor.

After finding a certain number of figures of the root, it will occur that the numbers to be added to the divisors will fall among the rejected figures, after which the remaining figures of the root will be found by simple division. It should be observed, however, that when places are rejected from the divisor, that whatever would have been carried had the complete multiplication been performed, is still to be carried to increase the last figure retained; and whenever the left-hand figure of those rejected, either in the dividend or divisor, exceeds 4, the last figure retained is to be increased by 1.

The following is the last example, wrought on the principle of admitting but three places of decimals into the dividends. The rejected figures, both in the dividends and divisors, are placed a little to the right.

6	
20	
26	
4	
50	
5.76	
55.76	
.16	
61.68	
.30	44
62.0	844
0	004
62.3	892
1	76325
62.4	

75.9	2.4257 +
52	
23.9	
22.304	
1.596	
1.240	688
.356	312
.312	827625
.044	484375
.043	
.001	

2. Find one root of the equation $x^3 + x^2 = 500$.

This equation is the same as $x^3 + x^2 + 0x = 500$; hence $b = 1$ $c = 0$, and $N = 500$.

The first figure of the root found by trial, is 7.

0	500 |7 . 61727975, &c., $= x$.
56	392
56	108
49	104 . 736
161	3 . 264
13 . 56	1 . 887181
174 . 56	1 . 376819
36	1 . 323862
188 . 48	. 052957
. 2381	. 037859
188 . 7181	. 015098
. 0001	. 013251
188 . 9563	. 001847
. 1669	. 001704
189 . 123|2	. 000143
189 . 290	. 000133
. 005	. 000010
. 1|8|9 . |2|9|5	. 000009

REMARK.—This example is wrought by retaining 5 decimal places in the dividends. We may *always* commence rejecting the places from the dividends, after having found 3 places of decimals in the root.

3. Find one root of the equation $x^3 - 17x^2 + 54x = 350$.

Ans. $x = 14 . 954$, &c.

4. Find one root of the equation $x^3 + 2x^2 + 3x = 13089030$.

Ans. $x = 235$.

5. Find one root of the equation $x^3 + 2x^2 - 23x = 70$.

Ans. $x = 5 . 1345$, &c.

REMARK.—In the preceding solutions only one root has been obtained, yet the others may be found with equal facility, by finding by trial the first figure in each and then proceeding by the rule already given. There is, however, a shorter method for determining the remaining roots.

Subtract the root found, taken with a contrary sign, from the coefficient of the second term of the given equation, and denote the remainder by a. Divide the absolute term by the root found, and denote the quotient by b; then will the roots of the equation

$$x^2 + ax + b = 0$$

be the two remaining roots of the given equation.

Method of resolving Higher Equations.

345. The general method of resolving cubic equations, has been explained in Art. 342. We shall now add from Young's Algebra, the method of resolving equations of a higher degree. It has not been thought best to give the general investigation, but merely to add, for the solution of an equation of any degree, the following general

RULE.

I. *Transpose the absolute term to the second member of the equation. Then, beginning with the co-efficient of the first term, arrange the co-efficients of the first member in a row, placing the absolute term to the right.*

II. *Having found the first figure of the root, multiply it by the first co-efficient and add the product to the second co-efficient; then multiply the sum by the same figure of the root and add the product to the third co-efficient; and so on to the last co-efficient: the last sum will be the first divisor, which multiply by the figure of the root and subtract the product from the absolute term: the result will be the second dividend.*

III. *Perform the same operations on the first co-efficient and the set of sums found, as was performed with the co-efficients, and the last sum will be the* TRIAL DIVISOR *for the second figure of the root. Then perform the same operations on the first co-efficient and the second set of sums, only stop in the column of the last co-efficient but one. Repeat the same operation on the first co-efficient and the last set of sums, but stop in the next left-hand column, and so on until you stop in the column of the second co-efficient.*

IV. *Then find from the trial divisor the second figure of the root, taking care that it be not too large. Take this second figure, and perform with it on the first co-efficient and the last set of sums the same operations as were performed on the co-efficients with the first figure of the root and the sum; in the last column will be the second divisor, which multiply by the second figure of the root and subtract the product from the second dividend.*

V. *The next trial divisor, the next figure of the root, and the true divisor, are found by the principles already explained, and the places of figures in the root may be carried as far as necessary.*